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REDUCED NORMAL FORM OF THE PERIODIC HAMILTONIAN SYSTEM

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Abstract

First we remind the normal form near a stationary solution of an autonomous Hamiltonian system. Second we consider the linear periodic Hamiltonian systems. For them we find normal forms of Hamiltonian functions in both complex and real cases. The real case has a specifficy in the case of parametric resonance. Then we find normal forms of the Hamiltonian functions for nonlinear periodic systems. By means of additional canonical transformation of coordinates, such system always is reduced to an autonomous Hamiltonian system, which preserves all small parameters and symmetries of the initial system. Its local families of stationary points correspond to families of periodic solutions of the initial system.

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1. Introduction

The resonant normal form of autonomous Hamiltonian system near a stationary solution, taking in account only eigenvalues of the matrix A of its linear part and without any restriction on the matrix A, was introduced in [1], Section 12. Appeared, that it is equivalent to a Hamiltonian system with a smaller number of degree of freedom.

Later it was introduced a more simple ultraresonant normal form which takes in account the Jordan blocks of the matrix A [2]. But these additional simplifications do not allow additionally to reduce the number of degrees of freedom.

Theory of the resonant normal form was given in details in Chapter I of [3] and here it is shortly remind in Section 2. The analogous theory of resonant normal form for periodic Hamiltonian system was given in Chapter II of [3]. However there are two defects:

● the case of parametric resonance was given not so good and;

● the normal form is not reduced to an autonomous system.

Here we correct these defects in Sections 3 and 4 correspondingly.

2. Normal Form of the Autonomous Hamiltonian System [3, Chapter I]

Let us consider the Hamiltonian system

$$\dot{\xi}_j = \frac{\partial \gamma}{\partial \eta_j}, \quad \dot{\eta}_j = -\frac{\partial \gamma}{\partial \xi_j}, \quad j = 1, \dots, n, \tag{1}$$

with n degrees of freedom in a vicinity of the stationary solution

$$\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) = 0, \quad \boldsymbol{\eta} = (\eta_1, \dots, \eta_n) = 0. \tag{2}$$

If the Hamiltonian function $\gamma(\boldsymbol{\xi}, \boldsymbol{\eta})$ is analytic in the point (2), then it is expanded into the power series

$$\gamma(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum \gamma_{\mathbf{pq}} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}}, \tag{3}$$

where $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n$, $\mathbf{p}, \mathbf{q} \geqslant 0$, $\xi^{\mathbf{p}} = \xi_1^{p_1} \xi_2^{p_2} \dots \xi_n^{p_n}$. Here $\gamma_{\mathbf{pq}}$ are constant coefficients. As the point (2) is stationary, then the expansion (3) begins from quadratic terms. They correspond to the linear part of the system (1). Eigenvalues of its matrix are decomposed in pairs:

$$\lambda_{j+n} = -\lambda_j, \quad j = 1, \dots, n.$$

Let $\lambda = (\lambda_1, \dots, \lambda_n)$. The canonical changes of coordinates

$$(\xi, \eta) \to (\mathbf{x}, \mathbf{y}) \tag{4}$$

preserve the Hamiltonian structure of the system.

Here

$$\mathbf{x} = (x_1, \dots, x_n),\ \mathbf{y} = (y_1, \dots, y_n).$$

Theorem 1. *There exists a formal canonical transformation* (4), *bringing the system* (1) *to the normal form*

$$\dot{x}_j = \frac{\partial g}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial g}{\partial x_j}, \quad j = 1, \dots, n, \tag{5}$$

where the series

$$g(\mathbf{x}, \mathbf{y}) = \sum g_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} \tag{6}$$

contains only terms with $\langle \mathbf{p} - \mathbf{q}, \lambda \rangle = 0$, *and the square part* $g_2(\mathbf{x}, \mathbf{y})$ *has its own normal form* (*i.e., the matrix of the linear part of the system is the Hamiltonian analog of the Jordan normal form*).

Here $\langle \mathbf{p}, \lambda \rangle = p_1\lambda_1 + \dots + p_n\lambda_n$ is the scalar product.

If $\lambda \neq 0$, then the normal form (5) is equivalent to a system with smaller number of degrees of freedom and with additional parameters. The normalizing transformation (4) conserves small parameters and linear automorphisms

$$(\xi, \eta) \to (\tilde{\xi}, \tilde{\eta}), \quad t \to \tilde{t}.$$

Local families (i.e., coming through the point (2)) of periodic solutions of systems (5), (6) satisfy the system of equations

$$\frac{\partial g}{\partial y_j} = \lambda_j x_j a, \quad \frac{\partial g}{\partial x_j} = \lambda_j y_j a, \quad j = 1, \dots, n,$$

where a is a free parameter. For the real initial system (1), the coefficients $g_{\mathbf{pq}}$ of the complex normal form (6) satisfy to special properties of reality and after a standard canonical linear change of coordinates $(\mathbf{x}, \mathbf{y}) \to (\mathbf{X}, \mathbf{Y})$ the system (5) transforms into a real system. There are several methods of computation of coefficients $g_{\mathbf{pq}}$ of the normal form (6). The most simple method was described in the book [4].

3. Normalization of a Linear Hamiltonian System

3.1. Linear system

We consider the linear system

$$\frac{d\zeta}{d\psi} = A(\psi)\zeta, \tag{7}$$

where the vector $\zeta = (\zeta_1, \dots, \zeta_m)$, $A(\psi)$ is a matrix depending of ψ analytically. After the change of coordinates

$$\zeta = B(\psi)\mathbf{z} \tag{8}$$

system (7) goes to the system

$$\frac{d\mathbf{z}}{d\psi} = B^{-1}\left(AB - \frac{dB}{d\psi}\right)\mathbf{z}. \tag{9}$$

Let now the system (7) be Hamiltonian system

$$\frac{d\xi_j}{d\psi} = \frac{\partial\gamma}{\partial\eta_j}, \quad \frac{d\eta_j}{d\psi} = -\frac{\partial\gamma}{\partial\xi_j}, \quad j = 1, \dots, n, \tag{10}$$

i.e., $m = 2n$, $\zeta = (\xi, \eta) = (\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n)$, $A(\psi) = J\Gamma(\psi)$, where $\Gamma(\psi)$ is a symmetric matrix, $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$ and the Hamiltonian function $\gamma = \frac{1}{2}\langle\zeta, \Gamma(\psi)\zeta\rangle$. Here E_n is the identical $n \times n$ matrix, and $\langle\cdot, \cdot\rangle$ is the scalar product. If the transformation (8) is canonical, i.e.,

$$B^*(\psi)JB(\psi) = \delta J, \quad \delta = \text{const.} \tag{11}$$

(star is the symbol of transposition of a matrix), then system (9) is also a Hamiltonian system and

$$g = \frac{1}{2\delta}\langle \mathbf{z}, B^*\Gamma B\mathbf{z}\rangle + \frac{1}{2\delta}\left\langle \mathbf{z}, B^*J\frac{dB}{d\psi}\mathbf{z}\right\rangle \overset{\text{def}}{=} \frac{1}{2}\langle \mathbf{z}, G\mathbf{z}\rangle, \tag{12}$$

i.e., $G = \delta^{-1}B^*\Gamma B + \delta^{-1}B^*JdB/d\psi$, $\mathbf{z} = (\mathbf{x}, \mathbf{y})$.

Now we consider the Hamiltonian system (10), where the matrix $A(\psi) = J\Gamma(\psi)$ has in ψ period 2π, i.e., $A(\psi + 2\pi) = A(\psi)$. We will try to obtain Hamiltonian (12) of the most simple form by means of linear canonical change of coordinates (8). Let $Z(\psi)$ be the fundamental matrix of solutions to system (7). Then

$$Z(\psi + 2\pi) = Z(\psi)N,$$

where N is a constant matrix, $\det N \neq 0$. It is canonical for Hamiltonian system. If matrix N can be written in the form

$$N = \exp(2\pi JL), \tag{13}$$

where L is a constant symmetrical matrix, then $L = B_1^*GB_1$ according to Section 1 of Chapter I [3], where B_1 is a constant canonical matrix and G is the normal form of matrix L. So transformation (8) with $B(\psi) = Z(\psi)\exp(-\psi JG)$ reduces Hamiltonian system (10) to normal form

$$\frac{d\mathbf{z}}{d\psi} = JG\mathbf{z}, \quad G = \text{const.}, \tag{14}$$

with Hamiltonian function $g = \frac{1}{2}\langle \mathbf{z}, G\mathbf{z}\rangle$. But writing (13) exists not for any canonical matrix N (see Williamson [5]). Let $\nu_1, \ldots, \nu_{2n}$ be the eigenvalues of the matrix N. Together with number $\nu_j = b$, they have the number b^{-1}. Moreover elementary divisors of the matrix $\nu E - N$ have following properties:

• if $b \neq \pm 1$ and there are k elementary divisors $(\nu - b)^l$, then there are exactly k elementary divisors $\left(\nu - b^{-1}\right)^l$;

• if $b = \pm 1$ and l is odd, then the elementary divisor $(\nu - b)^l$ presents even number times.

3.2. Complex normal form

For the complex system (7), the matrix N is also complex. Nonreduced over field of complex numbers $\mathbb{C}$ elementary divisors of the matrix $\nu E - N$ belong to one of the following cases:

(**C1**) $(\nu - b)^l$ and $\left(\nu - b^{-1}\right)^l$, $b \neq \pm 1$;

(**C2**) $(\nu - b)^l$ and $(\nu - b)^l$, $b = \pm 1$, l is odd;

(**C3**) $(\nu - 1)^{2l}$;

(**C4**) $(\nu + 1)^{2l}$.

By means of a constant canonical change of coordinates ζ, matrix $\Gamma(\psi)$ can be transformed to a such block form, that each mentioned cases corresponds to its own four blocks of dimension l, and zeros stay out of these blocks. So it is enough to consider each of these cases, assuming $l = n$. In cases (**C1**) – (**C3**), there is writing (13); here elementary divisors

$(\lambda - a)^l$ of the matrix $\lambda E - JL$ belong to the cases (**C1**) – (**C3**) of Subsection 1.2 of Chapter I [3], where

$$a = \frac{1}{2\pi}\operatorname{Ln} b = \frac{1}{2\pi}\ln|b| + \frac{i}{i\pi}\arg b + im,$$

and m is arbitrary integer; namely: in case (**C1**)

$$G = \begin{pmatrix} 0 & C^* \\ C & 0 \end{pmatrix},$$

where C is the Jordan block $l \times l$:

$$\begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 & 0 \\ \varepsilon & a & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \varepsilon & a & 0 \\ 0 & 0 & 0 & \dots & 0 & \varepsilon & a \end{pmatrix},$$

i.e.,

$$g_2 = a\sum_{j=1}^{l} x_j y_j + \varepsilon\sum_{j=1}^{l-1} x_j y_{j+1}; \tag{15}$$

case (**C2**) with $b = 1$ belongs to case (**C1**) with $a = im$; case (**C2**) with $b = -1$ belongs to case (**C1**) with $a = im + \frac{i}{2}$; in case (**C3**)

$$G = \begin{pmatrix} 0 & C^* \\ C & \sigma\Delta \end{pmatrix},$$

where C is the Jordan block of order l with $a = 0$, $\sigma = \pm 1$ and diagonal matrix $\Delta = \{1, 0, \dots, 0\}$, i.e.,

$$g_2 = \varepsilon\sum_{j=1}^{l-1} x_j y_{j+1} + \frac{1}{2}\sigma y_1^2; \tag{16}$$

in case (**C4**) writing (13) is absent and the complex normal form

$$\frac{d\mathbf{z}}{d\psi} = JG(\psi)\mathbf{z}$$

has

$$G = \begin{pmatrix} 0 & C^* \\ C & \sigma\Delta\exp(i\psi) \end{pmatrix},$$

where C is the Jordan block of order l with $a = im + \frac{i}{2}$, $\sigma = \pm 1$ [6], i.e.,

$$g_2 = a\sum_{j=1}^{l} x_j y_j + \varepsilon\sum_{j=1}^{l-1} x_j y_{j+1} + \frac{1}{2}\sigma y_1^2 \exp(i\psi). \tag{17}$$

Let us consider the double cases (**C3**) and (**C4**).

(**C3***) Two elementary divisors $(\nu - 1)^{2l'}$ and $(\nu - 1)^{2l'}$ correspond to normal form (14) of the case (**C2**) with $a = im$ and with arbitrary integral m (only now $l = 2l'$ is even).

(**C4***) Two elementary divisors $(\nu + 1)^{2l'}$ and $(\nu + 1)^{2l'}$ correspond to normal form of case (**C1**) with

$$a = (2\pi)^{-1}\mathrm{Ln}(-1) = im + \frac{i}{2}.$$

Thus, by the complex change (8), where $B(\psi)$ is canonical 2π-periodic matrix, the initial Hamiltonian function

$$\gamma = \frac{1}{2}\langle \zeta, \Gamma(\psi)\zeta \rangle$$

is reduced to a normal form, which is a sum of forms (15), (16), and (17). It is constant, if each elementary divisor $(\nu + 1)^{2l}$ presents even number times among elementary divisors of matrix $\nu E - N$. Williamson [5, Theorem 1] proved, that the condition is necessary and sufficient for complex reducibility.

3.3. Real system

For a real system (10), the matrix N is real. So elementary divisors of matrix $\nu E - N$ have following properties. Let elementary divisor $(\nu - b)^l$ present exactly k times.

• If number b is complex, i.e., $\operatorname{Re} b \cdot \operatorname{Im} b \neq 0$, and $|b| \neq 1$, then elementary divisors $(\nu - \overline{b})^l$, $(\nu - b^{-1})^l$ and $(\nu - \overline{b}^{-1})^l$ present also exactly k times.

• If number b is real or $|b| = 1$, $b \neq \pm 1$, then $(\nu - b^{-1})^l$ presents exactly k times.

• If $b = \pm 1$ and l is odd, then k must be even.

So elementary divisors of matrix $\nu E - N$ belong to one of following eight cases:

(**R1**) $(\nu - b)^l(\nu - \overline{b})^l$ and $(\nu - b^{-1})^l(\nu - \overline{b}^{-1})^l$, $b \in \mathbb{C}$, $\operatorname{Re} b \cdot \operatorname{Im} b \neq 0$, $|b| \neq 1$;

(**R2**) $(\nu - b)^l$ and $(\nu - b^{-1})^l$, $b \in \mathbb{R}$, $b > 0$, $b \neq 1$;

(**R3**) $(\nu - b)^l(\nu - \overline{b})^l$, $|b| = 1$, $b \neq \pm 1$;

(**R4**) $(\nu - 1)^l$ and $(\nu - 1)^l$, l is odd;

(**R5**) $(\nu - 1)^{2l}$;

(**R6**) $(\nu + 1)^l$ and $(\nu + 1)^l$, l is odd;

(**R7**) $(\nu - b)^l$ and $(\nu - b^{-1})^l$, $b \in \mathbb{R}$, $b < 0$, $b \neq -1$;

(**R8**) $(\nu + 1)^{2l}$.

By means of a real constant canonical change of coordinates, matrix $\Gamma(\psi)$ can be transformed to a such block form, that each of mentioned cases corresponds to its own four blocks of dimension l and zeros stay out of these blocks. So it is enough to consider each of these cases, assuming $l = n$. In cases (**R1**) – (**R7**) there exists writing (13) with real matrix L; here elementary divisors $(\lambda - a)^l$ of matrix $\lambda E - JL$ belong to cases (**R1**) – (**R5**) of Subsection 1.3 of Chapter I [3], where

$$a = \frac{1}{2\pi}\,\mathrm{Ln}\, b = \frac{1}{2\pi}\ln b + im.$$

Here the number $\ln b$ is uniquely determined by number b, but the integer m must be calculated by the following method. We compute any solution to a linear subsystem of form (7) belonging to a case among (**R1**) – (**R7**). Number of oscillations each of its coordinates in period 2π is the number m. If we make the additional transformation

$$\tilde{x}_j = x_j \exp(-im\psi), \quad \tilde{y}_j = y_j \exp(im\psi), \quad j = 1, \ldots, l,$$

then we obtain the eigenvalue $\tilde{\lambda} : 0 \leqslant \mathrm{Im}\,\tilde{\lambda} \leqslant 1$. Here in cases (**R3**) and (**R5**) there is an additional real invariant $\sigma = \pm 1$. Thus, in cases (**R1**) – (**R7**), there is constant complex normal form of Hamiltonian function

$$g_2 = \frac{1}{2}\langle \mathbf{z}, G\mathbf{z}\rangle,$$

that can be translated into real normal form

$$f_2 = \frac{1}{2}\langle \mathbf{Z}, F\mathbf{Z}\rangle$$

by means of the standard canonical transformation

$$\mathbf{Z} = Q\mathbf{z}, \quad \det Q = 1.$$

Here substitution

$$\overline{\mathbf{z}}, P\mathbf{z},$$

where $2n$-matrix $P = \overline{Q}^{-1}Q$, preserves Hamiltonian function. Concrete form of matrices Q and P for each of cases (**R1**) – (**R7**) is described in Chapter I [3]. So in cases (**R2**) – (**R7**) either

$$x_j = X_j = \widetilde{x}_j, \quad y_j = Y_j = \overline{y}_j, \quad j = 1, \dots, l, \tag{18}$$

or

$$x_j = \frac{1}{\sqrt{2i}}(iX_j - Y_j) = i\overline{y}_j,\ y_j = \frac{1}{\sqrt{2i}}(iX_j + Y_j) = i\overline{x}_j,\ j = 1, \dots, l. \tag{19}$$

Theorem 2. *Complex writing of the real normal form in case* (**R8**) *is system* (17) *with such standard transformation*

$$x_j = \frac{1}{2i}\{X_j[1 + i\exp(-it)] - Y_j[i + \exp(-it)]\},$$

$$y_j = \frac{1}{2i}\{X_j[i - \exp(it)] + Y_j[-1 + i\exp(it)]\}. \tag{20}$$

Here $\overline{x}_j = ix_je^{it}$, $\overline{y}_j = -iy_je^{-it}$, $\varepsilon = i$, $\sigma = \pm i$, $j = 1, \dots, l$.

Then

$$\overline{G}(\psi) = G(\psi),$$

and Hamiltonian function of the normal form (17) is

$$g_2 = \lambda\sum_{j=1}^{l} x_jy_j + i\sum_{j=1}^{l-1} x_jy_{j+1}$$

$$\pm \frac{i}{2}(X_1^2 + Y_1^2 + X_1^2 \sin\psi - 2X_1Y_1\cos\psi - Y_1^2 \sin\psi).$$

4. Nonlinear Normal Form

4.1. Nonlinear normalization

We consider the Hamiltonian system with n degrees of freedom

$$\frac{d\xi_j}{d\psi} = \frac{d\gamma}{\partial\eta_j}, \quad \frac{d\eta_j}{d\psi} = -\frac{\partial\gamma}{\partial\xi_j}, \quad j = 1, \dots, n, \tag{21}$$

where γ is a power series in $\boldsymbol{\xi}, \boldsymbol{\eta}$ with 2π-periodic in ψ coefficients, which is expanded into a convergent Poisson series

$$\gamma = \sum_m \gamma_{\mathbf{pq}m} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}} \exp(im\psi), \tag{22}$$

beginning from quadratic form g_2 in $\boldsymbol{\xi}, \boldsymbol{\eta}$. We make the linear canonical transformation $\boldsymbol{\xi}, \boldsymbol{\eta} \to \mathbf{x}, \mathbf{y}$, which is 2π-periodic in ψ and brings the square part of Hamiltonian function of system (21) to complex normal form being a sum of parts (15), (16), (17). Then on the main diagonal of matrix JG are its eigenvalues $\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n$. Denote $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Hamiltonian (22) takes the form

$$g(\mathbf{x}, \mathbf{y}, \psi) = \sum g_{\mathbf{pq}m} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} \exp(im\psi). \tag{23}$$

We call it as *normal form*, if

(1) its form g_2 is normal form (15), (16), (17);

(2) expansion (22) contains only resonant terms, for which

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle + im = 0. \tag{24}$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product.

Theorem 3. *For the Hamiltonian* (23), *there exists the* 2π*-periodic in* φ *formal canonical change of coordinates* $\mathbf{x}, \mathbf{y}, \psi \to \mathbf{u}, \mathbf{v}, \varphi$:

$$\mathbf{z} = (\mathbf{x}, \mathbf{y}) = \mathbf{w} + \mathbf{b}(\mathbf{w}, \varphi), \; \psi = \varphi + b_{2n+1}(\mathbf{w}, \varphi),$$

which transforms Hamiltonian (23) *into normal form*

$$h(\mathbf{u}, \mathbf{v}, \varphi) = \sum h_{\mathbf{pq}m} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}} \exp(im\varphi), \tag{25}$$

with property (24). *Here* $\mathbf{w} = (\mathbf{u}, \mathbf{v}) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$.

Proof see in Chapters I and II [3].

Theorem 4. *The canonical transformation*

$$u_j = \tilde{u}_j \exp(-i \operatorname{Im} \lambda_j \varphi), \quad v_j = \tilde{v}_j \exp(i \operatorname{Im} \lambda_j \varphi), \quad j = 1, \ldots, n,$$

reduces the Hamiltonian normal form (25) *to constant power series*

$$\tilde{h}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \sum \tilde{h}_{\mathbf{pq}m} \tilde{\mathbf{u}}^{\mathbf{p}} \tilde{\mathbf{v}}^{\mathbf{q}}, \tag{26}$$

where

$$\langle \mathbf{p} - \mathbf{q}, \operatorname{Re} \lambda \rangle = 0, \quad \langle \mathbf{p} - \mathbf{q}, \operatorname{Im} \lambda \rangle = -m, \tag{27}$$

$\mathbf{p}, \mathbf{q}, m$ *are integer*, $\mathbf{p}, \mathbf{q} \geqslant 0$. *Here* $\tilde{h}_{\mathbf{pq}m} = h_{\mathbf{pq}m}$, *if* $\|\mathbf{p}\| + \|\mathbf{q}\| \geqslant 3$, *quadratic terms have form* $\tilde{h}_2(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \frac{1}{2}\left\langle \tilde{\mathbf{w}}, \tilde{G}\tilde{\mathbf{w}} \right\rangle$, *where matrix* $J\tilde{G} = JG - \operatorname{Im} \Lambda$ *with diagonal matrix* $\Lambda = \{\lambda, -\lambda\}$, *so quadratic terms in* $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \tilde{\mathbf{w}}$ *are almost absent.*

Proof is reduced to checking equality (11), which is evident here. Here $\|\mathbf{p}\| = p_1 + p_2 + \cdots + p_n$. For initial real Hamiltonian (22) complex coordinates $\mathbf{z}$ are connected with real coordinates $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$ by the standard transformation, formed by changes (18), (19), (20), and coordinates $\mathbf{w}$ and $\tilde{\mathbf{w}}$ are connected by the same changes with corresponding real coordinates $\mathbf{W}$ and $\tilde{\mathbf{W}}$ [3]. Thus, we come to autonomous Hamiltonian system with n degrees of freedom, which is named as *reduced normal form*.

4.2. Small parameters

Let the initial Hamiltonian is expanded into a power series in small parameters $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s)$. According to Theorem 5.1 of Chapter I [3], the normalizing transformation does not change small parameters. So we obtain the autonomous Hamiltonian (26), (27), where coefficients $\tilde{h}_{\mathbf{p}, \mathbf{q}, m}$ are power series in small parameters $\boldsymbol{\mu}$. For $\boldsymbol{\mu} = 0$ these coefficients with $\|\mathbf{p}\| + \|\mathbf{q}\| = 1$ equal to zero, but with $\|\mathbf{p}\| + \|\mathbf{q}\| = 2$ correspond to (27). However for $\boldsymbol{\mu} \neq 0$ it is not necessary.

For a system, corresponding to Hamiltonian (26), (27) we can compute families of stationary points near the point $\tilde{\mathbf{u}} = \tilde{\mathbf{v}} = 0$, $\boldsymbol{\mu} = 0$. It can be done by algorithms of power geometry [7]. Families of stationary points of the reduced normal form (26), (27) corresponds to families of periodic solutions to initial Hamiltonian system. Examples of such computations are in [8].

4.3. Linear canonical automorphisms

Let the initial system (21) have the linear canonical automorphism

$$\zeta^* = M\tilde{\zeta}^*, \quad \psi = \theta\tilde{\psi},$$

where M is constant $2n \times 2n$ matrix and $\theta = \text{const}$. According to Theorem 2.3 of Chapter I [3], the reduced normal form (26), (27) also has a corresponding linear canonical automorphism. However it can have additional automorphisms, which are absent in initial system [9].

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