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SOME GENERATION RESULTS OF INTEGRAL RESOLVENT FOR VOLTERRA EQUATIONS

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Abstract

The aim of this paper is to give an extension to a Favard classes for integral resolvent of a scalar Volterra integral equations similar to the one for semigroups and resolvent families (i.e., k-regularized resolvent family with $k(t) = 1$). In fact, we recover several well-known results for semigroups if we consider $a(t) = 1$ and for resolvent families if we consider $a(t)$ arbitrary and $k(t) = 1$.

1. Introduction

The Favard class for semigroups was developed in 1967 by Butzer and Berens presented in the monograph [5], for a very detailed reference, see in the monograph [10]. Applications appear in particular in [9, 21, 24], but are certainly not restricted to this. However, these concepts have

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been slightly introduced to Volterra integral equations in [14, 17, 3], in particular for the semigroup theory the Favard class plays an important role in localization and determination of the space of control operators p-admissible (resp., (p, q)-admissible) for well-posed linear (resp., bilinear) systems in [20, 4].

In this paper, we introduce the Favard spaces for a integral resolvent, extending some of the well-known theorems for semigroup and resolvent family.

In Section 2, we give some preliminaries about the concept of integral resolvent, and the relationship between linear integral equation of Volterra type with scalar kernel.

It is well-known that for a Cauchy problem there are strong relations connecting its semigroup solution and its associated generator. Likewise, for a Volterra scalar problem, there are some results connecting its integral resolvent family and the domain of the associated generator which will be reviewed in Section 2.

In Section 3, we define the Favard spaces for integral resolvent of a scalar Volterra integral equations, and for these spaces we account for some results which are similar to those of semigroups and resolvent families.

2. Preliminaries

In this section, we collect some elementary facts about scalar Volterra equations and integral resolvent. These topics have been covered in detail in [23, 17]. We refer to these works for reference to the literature and further information.

Let $(X, \|\cdot\|)$ be a Banach space, A be a linear closed densely defined operator in X and $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ is a scalar kernel, we consider the linear Volterra equation:

$$x(t) = \int_0^t a(t-s)Ax(s)ds + f(t), \quad t \geq 0, \tag{2.1}$$

where $f \in \mathcal{C}(\mathbb{R}^+, X)$.

We denote by $[D(A)]$ the domain of A equipped with the graph-norm. We define the convolution product $*$ of the scalar function a with the vector-valued function f by

$$(a * f)(t) := \int_0^t a(t-s)f(s)ds, \quad t \geq 0.$$

Definition 2.1 ([23]). A function $x \in \mathcal{C}(\mathbb{R}^+, X)$ is called:

(1) strong solution of (2.1) if $x \in \mathcal{C}(\mathbb{R}^+, [D(A)])$ and (2.1) is satisfied.

(2) mild solution of (2.1) if $a * x \in \mathcal{C}(\mathbb{R}^+, [D(A)])$ and

$$x = f(t) + A[a * x](t), \quad t \geq 0. \tag{2.2}$$

Obviously, every strong solution of (2.1) is a mild solution. Conditions under which mild solutions are strong solutions are studied in [23].

Definition 2.2 ([23]). Equation (2.1) is called well-posed if for each $\upsilon \in D(A)$, there is a unique strong solution $x(t, \upsilon)$ on $\mathbb{R}^+$ of

$$x(t, \upsilon) = a(t)\upsilon + (a * Ax)(t), \quad t \geq 0, \tag{2.3}$$

and for a sequence $(\upsilon_n) \subset D(A)$, $\upsilon_n \to 0$ implies $x(t, \upsilon_n) \to 0$ in X uniformly on compact intervals.

Definition 2.3 ([23]). Let $a \in \mathcal{C}(\mathbb{R}^+)$ be a scalar kernel. A strongly continuous family $(R(t))_{t \geq 0} \subset \mathcal{L}(X)$; (the space of bounded linear operators in X) is called an integral resolvent for Equation (2.1), if the following three conditions are satisfied:

(R1) $R(0) = a(0)I$.

(R2) $R(t)$ commutes with A, which means $R(t)(D(A)) \subset D(A)$ for all $t \geq 0$, and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

(R3) For each $x \in D(A)$ and all $t \geq 0$, the resolvent equations hold:

$$R(t)x = a(t)x + \int_0^t a(t-s)AR(s)xds.$$

Note that the integral resolvent for (2.1) is uniquely determined and further information on an integral resolvent can be found in [17].

We also notice that when $a(t) = 1$, then $R(t)$ corresponds to a C_0-semigroup.

If there exists an integral resolvent for (2.1), then a mild solution of (2.1) may be obtained by the formula

$$x(t) = f(t) + A\int_0^t R(t-s)f(s)ds, \quad t \geq 0.$$

Suppose $R(t)$ is an integral resolvent for (2.1), let $f \in \mathcal{C}(\mathbb{R}^+, X)$ and $x \in \mathcal{C}(\mathbb{R}^+, X)$ be a mild solution for (2.1). Then $R * x$ is well-defined and continuous and we obtain from (R3) and (2.1)

$$a * x = (R - Aa * R) * x = R * x - R * Aa * x = R * f,$$

hence $R * f \in \mathcal{C}(\mathbb{R}^+, [D(A)])$ and from (2.1), we obtain

$$x(t) = f(t) + A\int_0^t R(t-s)f(s)ds, \quad t \geq 0.$$

The following result establishes the relation between well-posedness and existence of an integral resolvent.

In what follows, $\mathcal{R}$ denotes the range of a given operator.

Lemma 2.4 ([17, Theorem 2.4])**.** (2.1) *is well-posed if and only if* (2.1) *admits an integral resolvent* $(R(t))_{t \geq 0}$. *If this is the case, we have in addition* $\mathcal{R}(a * R(t)) \subset D(A)$, *for all* $t \geq 0$ *and*

$$R(t)x = a(t)x + A\int_0^t a(t-s)R(s)xds, \tag{2.4}$$

for each $x \in X$, $t \geq 0$.

From this we obtain that if $(R(t))_{t\geq 0}$ is integral resolvent of (2.1), we have $A(a * R)(\cdot)$ is strongly continuous.

Remark 2.5. Recall from [23, Chapter 1] that given $a \in L^1_{\text{loc}}(\mathbb{R}^+)$, a strongly continuous family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$ is called resolvent family for Equation (2.1), if the following three conditions are satisfied:

(S1) $S(0) = I$.

(S2) $S(t)$ commutes with A, which means $S(t)D(A) \subset D(A)$ for all $t \geq 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

(S3) For each $x \in D(A)$ and all $t \geq 0$, the resolvent equations hold:

$$S(t)x = x + \int_0^t a(t-s)AS(s)x ds.$$

The importance of the resolvent family $S(t)$ is that, if it exists, then the solution $x(t)$ of (2.1) is given by the following variation of parameters formula in [23]:

$$x(t) = \frac{d}{dt}\int_0^t S(t-s)f(s)ds, \tag{2.5}$$

for all $t \geq 0$, and

$$x(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds, \tag{2.6}$$

where $t \geq 0$ and $f \in W^{1,1}(\mathbb{R}^+, X)$, gives us a mild solution for (2.1).

Definition 2.6 ([8]).

− An integral resolvent $(R(t))_{t\geq 0}$ is called exponentially bounded, if there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|R(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and the pair (M, ω) is called type of $(R(t))_{t\geq 0}$.

– The growth bound of $(R(t))_{t\geq 0}$ is $\omega_0 = \inf\{\omega \in \mathbb{R}, \|R(t)\| \leq Me^{\omega t}, t \geq 0, M > 0\}$, if $\omega_0 < 0$ the integral resolvent is called exponentially stable.

Note that, contrary to the case of C_0-semigroup, an integral resolvent for (2.1) need not to be exponentially bounded (see [8, 23, 17]). However, there is checkable conditions guaranteeing that (2.1) possesses an exponentially bounded integral resolvent.

We will use the Laplace transform at times, suppose $g : \mathbb{R}^+ \to X$ is measurable and there exist $M > 0, \omega \in \mathbb{R}$, such that $\|g(t)\| \leq Me^{\omega t}$ for almost $t \geq 0$, then the Laplace transform

$$\hat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\, dt,$$

exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$.

A function $a \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ is ω (resp., ω^+)-exponentially bounded if $\int_0^\infty e^{-\omega s}|a(s)|\,ds < +\infty$ for some $\omega \in \mathbb{R}$ (resp., $\omega > 0$).

The following proposition stated in [23, Theorem 1.4], establishes the relation between an integral resolvent and Laplace transform.

Proposition 2.7. *Let* $a \in \mathcal{C}(\mathbb{R}^+)$ *be* ω*-exponentially bounded then* (2.1) *admits an integral resolvent* $(R(t))_{t\geq 0}$ *of type* (M, ω) *if and only if the following conditions hold*:

(1) $\hat{a}(\lambda) \neq 0$ *and* $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$, *for all* $\lambda > \omega$ *where* $\rho(A)$ *is the set resolvent of* A.

(2) $K(\lambda) := \left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1}$ *called the resolvent associated to* $R(t)$ *satisfies*:

$$\sum_{n=0}^{+\infty} \frac{(\lambda - \omega)^{-(n+1)} \|K^{(n)}(\lambda)\|}{n!} \leq M \text{ for all } \lambda > \omega.$$

Under these assumptions, the Laplace-transform of $R(\cdot)$ *is well-defined and it is given by* $\widehat{R}(\lambda) = K(\lambda)$ *for all* $\lambda > \omega$.

Assuming the existence of an integral resolvent $(R(t))_{t \geq 0}$ for (2.1), it is natural to ask how to characterize the domain $D(A)$ of the operator A in terms of the integral resolvent. For very special case, the answer to the above question is well-known. For instance, when $a(t) = 1$ or $a(t) = t$, A is the generator of a C_0-semigroup $(\mathbb{T}(t))_{t \geq 0}$ or a cosine family $(C(t))_{t \geq 0}$ and we have:

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} \frac{1}{t} (\mathbb{T}(t)x - x) = Ax\right\},$$

and

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} \frac{2}{t^2} (C(t)x - x) = Ax\right\},$$

respectively (see [25, 10]).

It was proved in [19, 17] if $|a(t)|$ is continuous and nondescreasing and

$$\limsup_{t \mapsto 0^+} \frac{\|R(t)\|}{|a(t)|} < \infty,$$

that:

$$D(A) = \{x \in X \,/\, \lim_{t \to 0^+} \frac{R(t)x - a(t)x}{(a * a)(t)} = Ax\}. \tag{2.7}$$

From now and in view of this result we say that the pair (A, a) is a generator of an integral resolvent $(R(t))_{t \geq 0}$.

3. Favard Spaces for an Integral Resolvent with Kernel

The following definition which corresponds to a natural extension, in our context, of the Favard class frequently used in approximation theory for semigroups and resolvent families (see, e.g., [21, 10, 17, 3]).

Definition 3.1. Let (2.1) admits a bounded integral resolvent $(R(t))_{t\geq 0}$ on X, for ω^+-exponentially bounded $a \in L^1_{\text{loc}}(\mathbb{R}^+)$. For $0 < \alpha \leq 1$, we define the Favard spaces (frequency and temporal) of order α associated to (A, a) as follows:

$$F^{\alpha}(A) := \left\{x \in X / \sup_{\lambda > \omega}\left\|\lambda^{\alpha} A\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} x\right\| < \infty\right\},$$

$$= \left\{x \in X / \sup_{\lambda > \omega}\|\lambda^{\alpha} AK(\lambda)x\| < \infty\right\},$$

and

$$\widetilde{F}^{\alpha}(A) := \left\{x \in X / \sup_{t>0} \frac{\|R(t)x - a(t)x\|}{|(a * a)(t)|^{\alpha}} < \infty\right\}.$$

Remark 3.2.

(i) When $a(t) = 1$, we recall that $(R(t))_{t\geq 0}$ corresponds to a bounded C_0-semigroup generated by A. In this situation we obtain

$$F^{\alpha}(A) = \left\{x \in X/\sup_{\lambda>0}\left\|\lambda^{\alpha} A(\lambda I - A)^{-1} x\right\| < \infty\right\},$$

and

$$\widetilde{F}^{\alpha}(A) := \left\{x \in X / \sup_{t>0} \frac{\|\mathbb{T}(t)x - x\|}{t^{\alpha}} < \infty\right\},$$

and we have $F^{\alpha}(A) = \widetilde{F}^{\alpha}(A)$. This case is well-known (see, e.g., [10]).

(ii) The Favard class of A with kernel $a(t)$ can be alternatively defined as the subspace of X given by

$$\left\{x \in X \,/\, \limsup_{\lambda\to\infty} \left\| \lambda^{\alpha} A \left(\frac{1}{\widehat{a}(\lambda)} I - A \right)^{-1} x \right\| < \infty \right\}.$$

As a consequence of $R(t)$ being bounded, the above space coincides with $F^{\alpha}(A)$ in Definition 3.1 and that

$$\widetilde{F}^{\alpha}(A) := \left\{ x \in X \,/\, \sup_{0<t\leq 1} \frac{\|R(t)x - a(t)x\|}{|(a * a)(t)|^{\alpha}} < \infty \right\}.$$

Proposition 3.3. *Let* $a = 1 + 1 * l$ *with* $l \in BV_{\text{loc}}(\mathbb{R}^{+})$; *the space of functions of locally bounded variation and* (A, a) *be a generator of a bounded integral resolvent* $(R(t))_{t\geq 0}$ *on X. In this case,*

$$\widetilde{F}^{\alpha}(A) = \left\{ x \in X \,/\, \sup_{0<t\leq 1} \frac{\|R(t)x - a(t)x\|}{t^{\alpha}} < \infty \right\}.$$

Proof. We write

$$\widetilde{F}^{\alpha}(A) = \left\{ x \in X \,/\, \sup_{0<t\leq 1} \frac{\|R(t)x - a(t)x\|}{|(a * a)(t)|^{\alpha}} < \infty \right\}$$

$$= \left\{ x \in X \,/\, \sup_{0<t\leq 1} \left(\frac{\|R(t)x - a(t)x\|}{t^{\alpha}} \times \frac{t^{\alpha}}{|(a * a)(t)|^{\alpha}} \right) < \infty \right\}$$

$$= \left\{ x \in X \,/\, \sup_{0<t\leq 1} \left(\frac{\|R(t)x - a(t)x\|}{t^{\alpha}} \times \left| \left(\frac{t}{(a * a)(t)} \right)^{\alpha} \right| \right) < \infty \right\}$$

$$= \left\{ x \in X \,/\, \sup_{0<t\leq 1} \left(\frac{\|R(t)x - a(t)x\|}{t^{\alpha}} \times \frac{1}{\left| \left(\frac{(a * a)(t)}{t} \right)^{\alpha} \right|} \right) < \infty \right\}$$

$$= \left\{ x \in X \,/\, \sup_{0<t\le 1} \frac{\|R(t)x - a(t)x\|}{t^{\alpha}} < \infty \right\}.$$

(due to $\lim_{t\to 0^+} \frac{(a * a)(t)}{t} = 1$). □

We prove that $F^{\alpha}(A)$ is stable by $R(t)$ for any scalar kernel a.

Proposition 3.4. *We have* $R(t)(F^{\alpha}(A)) \subset F^{\alpha}(A)$, *for all* $\alpha \in]0, 1]$ *and* $t \ge 0$.

Proof. For all $x \in D(A)$ and $t \ge 0$, we have by (R2):

$$AR(t)x = R(t)Ax,$$

then

$$\frac{1}{\hat{a}(\lambda)} R(t) - AR(t) = \frac{1}{\hat{a}(\lambda)} R(t) - R(t)A,$$

i.e.,

$$\left(\frac{1}{\hat{a}(\lambda)} I - A\right) R(t) = R(t)\left(\frac{1}{\hat{a}(\lambda)} I - A\right),$$

then under Proposition 2.7;

$$\hat{a}(\lambda) \ne 0 \text{ and } \frac{1}{\hat{a}(\lambda)} \in \rho(A),$$

hence we have

$$\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1} R(t) = R(t)\left(\frac{1}{\hat{a}(\lambda)} I - A\right)^{-1}. \tag{3.1}$$

Now, if $x \in F^{\alpha}(A)$, then

$$\sup_{\lambda > \omega} \left\| \lambda^{\alpha} A \left(\frac{1}{\hat{a}(\lambda)} I - A \right)^{-1} x \right\| < \infty,$$

by (3.1) and the boundedness of $R(t)$, we have

$$\sup_{\lambda>\omega}\left\|\lambda^{\alpha}A\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}R(t)\,x\right\| = \sup_{\lambda>\omega}\left\|\lambda^{\alpha}AR(t)\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}x\right\|$$

$$= \sup_{\lambda>\omega}\left\|\lambda^{\alpha}R(t)\,A\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}x\right\|$$

$$\leq \|R(t)\|\times\sup_{\lambda>\omega}\left\|\lambda^{\alpha}A\left(\frac{1}{\hat{a}(\lambda)}I-A\right)^{-1}x\right\|$$

$$< +\infty,$$

then $R(t)x \in F^{\alpha}(A)$ for all $t \geq 0$, hence we deduce that $R(t)(F^{\alpha}(A)) \subset F^{\alpha}(A)$, for all $\alpha \in]0, 1]$ and $t \geq 0$. $\square$

The proof of the following proposition is immediate.

Proposition 3.5. *The Favard classes of order* α *of* A *with kernel* $a(t)$, $F^{\alpha}(A)$ *and* $\widetilde{F}^{\alpha}(A)$ *are Banach spaces with respect to the norms*

$$\|x\|_{F^{\alpha}(A)} := \|x\| + \sup_{\lambda>\omega}\left\|\lambda^{\alpha}A\left(\frac{1}{\hat{a}(\lambda)}I - A\right)^{-1}x\right\|,$$

and

$$\|x\|_{\widetilde{F}^{\alpha}(A)} := \|x\| + \sup_{0<t\leq 1}\frac{\|R(t)x - a(t)x\|}{|(a * a)(t)|^{\alpha}},$$

respectively.

Without loss of generality we may assume that $\int_0^t |a(t-s)a(s)|^p\,ds \neq 0$ for all $t > 0$ and some $1 \leq p < \infty$. Otherwise, we would have for some $t_0 > 0$ and $p_0 \geq 1$ that $a(t) = 0$ for almost all $t \in [0, t_0]$: and thus by definition of an integral resolvent $R(t) = 0$ for $t \in [0, t_0]$. This implies that A is bounded, which is the trivial case with $X = D(A)$.

In what follows, we will use the following assumption on a such that

$$\int_0^t |a(t-s)a(s)|^p \, ds \neq 0,$$

for all $t > 0$ and some $1 \leq p < \infty$.

Assumption A1. There exist $\epsilon_a > 0$ and $t_a > 0$, such that for all $0 < t \leq t_a$, we have:

$$\left| \int_0^t a(t-s)a(s) ds \right| \geq \epsilon_a \int_0^t |a(t-s)a(s)|^p \, ds.$$

This is the case for functions a, which are positive (resp., $h(I) \subset]0, 1]$, with $h(t) = a(t-s)a(s)$, $s \in [0, t]$, $t \in I$) at some interval $I = [0, t_a[$ for $p = 1$ (resp., $p > 1$).

On the other hand, note that for all a; ω^+-exponentially bounded function, it is clear that $(a * a)^\alpha$ is also ω^+-exponentially bounded (due to $x^\alpha \leq 1 + x$ for $x \geq 0$ and $\alpha \in]0, 1]$).

We will consider the following assumption on $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ and $0 < \alpha \leq 1$.

Assumption A2. a is ω^+-exponentially bounded and there exists $\epsilon_{a,\alpha} > 0$, such that for all $\lambda > \omega$

$$|\hat{a}(\lambda)| \geq \epsilon_{a,\alpha} . \lambda^\alpha . \int_0^\infty e^{-\lambda t} |(a * a)(t)|^\alpha \, dt.$$

Example 3.6.

(i) The famous case $a(t) = 1$ satisfies the condition of Assumption A2 for all $\alpha \geq 0$ due to

$$\frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \cdot \int_0^{\infty} e^{-\lambda t} ((a * a)(t))^{\alpha} \, dt = \Gamma(\alpha + 1) \text{ for all } \lambda > 0,$$

which corresponds to the semigroup case (here Γ denotes the Gamma function).

(ii) Let $a(t) = t^{\beta}$, $-1 < \beta < 0$. We have $\hat{a}(\lambda) = \frac{\Gamma(\beta+1)}{\lambda^{\beta+1}}$ for all $\lambda > 0$, and $(a * a)(t) = \frac{(\Gamma(\beta+1))^2}{\Gamma(2\beta+1)} \cdot t^{2\beta+1}$ for all $t \geq 0$. Hence

$$\frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{\infty} e^{-\lambda t} ((a * a)(t))^{\alpha} \, dt = \lambda^{\alpha+\beta+1} \cdot \frac{(\Gamma(\beta+1)\,\Gamma(\beta+1))^{\alpha}}{\Gamma(\beta+1)\,(\Gamma(2\beta+1))^{\alpha}} \int_0^{\infty} e^{-\lambda t} t^{\alpha(2\beta+1)} dt,$$

$$= \lambda^{\beta - 2\alpha\beta} \cdot \frac{(\Gamma(\beta+1))^{\alpha}\,\Gamma(2\alpha\beta + \alpha + 1)}{(\Gamma(\beta+1))^{1-\alpha}\,(\Gamma(2\beta+1))^{\alpha}}.$$

Then a satisfy Assumption A2 for $-1 < \beta < 0$ and $0 < \alpha < \frac{1}{2}$.

The following result establishes the relation between the spaces $\widetilde{F}^{\alpha}(A)$ and $F^{\alpha}(A)$ which is similar to [10, Proposition 5.12] and [3, Proposition 4.7].

Proposition 3.7. *Let* (2.1) *admits a bounded integral resolvent* $(R(t))_{t \geq 0}$ *on* X, *for* ω^{+}*-exponentially bounded* $a \in L^1_{\text{loc}}(\mathbb{R}^{+})$ *and* $0 < \alpha \leq 1$. *Assume that there is a constant* $N > 0$ *such that* $1 \leq a(t) \leq N$ *for all* $t \geq 0$.

(i) *If a satisfies Assumption A*1 *with* $p = 1$, *then* $F^{\alpha}(A) \subset \widetilde{F}^{\alpha}(A)$.

(ii) *If a is non negative satisfying Assumption A*2, *then* $\widetilde{F}^{\alpha}(A) \subset F^{\alpha}(A)$.

Proof.

(i) Let $x \in F^{\alpha}(A)$ and $0 < t \leq 1$. Then $\sup_{\lambda > \omega} \|\lambda^{\alpha} AK(\lambda)x\| =: K_x < \infty$. Using the integral representation of the resolvent (see Proposition 2.7), we obtain

$$x = \frac{1}{\hat{a}(\lambda)} K(\lambda)x - AK(\lambda)x \text{ for } \lambda > \omega$$

$$=: x_{\lambda} - y_{\lambda}.$$

Since $x_{\lambda} \in D(A)$ and using (R2) – (R3), we have

$$\begin{aligned}
\|R(t)x_{\lambda} - a(t)x_{\lambda}\| &= \left\| \int_0^t a(t-s)R(s)Ax_{\lambda} ds \right\| \\
&\leq \int_0^t |a(t-s)| \cdot \|R(s)\| \cdot \|Ax_{\lambda}\| ds \\
&\leq M \cdot \|Ax_{\lambda}\| \cdot \int_0^t |a(s)| ds \\
&= M \cdot \| \lambda^{\alpha} AK(\lambda)x \| \cdot \left| \frac{1}{\lambda^{\alpha} \hat{a}(\lambda)} \right| \cdot (1 * |a|)(t) \\
&\leq MK_x \cdot \left| \frac{1}{\lambda^{\alpha} \hat{a}(\lambda)} \right| \cdot (1 * |a|)(t).
\end{aligned}$$

On the other hand, $(R(t))_{t \geq 0}$ and $a(t)$ are bounded by M and N, respectively, we have

$$\begin{aligned}
\|R(t)y_{\lambda} - a(t)y_{\lambda}\| &\leq \|R(t)y_{\lambda}\| + \|a(t)y_{\lambda}\| \\
&\leq \|R(t)\| . \|y_{\lambda}\| + |a(t)| \|y_{\lambda}\| \\
&\leq (M + N) . \|y_{\lambda}\| \\
&= (M + N) . \|AK(\lambda)x\|
\end{aligned}$$

$$= (M + N).\left|\frac{1}{\lambda^{\alpha}}\right|.\|\lambda^{\alpha} AK(\lambda)x\|$$

$$\leq (M + N)K_x \cdot \left|\frac{1}{\lambda^{\alpha}}\right|.$$

This implies

$$\frac{\|R(t)x - a(t)x\|}{|(a*a)(t)|^{\alpha}} \leq \frac{MK_x \cdot \left|\frac{1}{\lambda^{\alpha}\hat{a}(\lambda)}\right| \cdot (1*|a|)(t)}{|(a*a)(t)|^{\alpha}} + \frac{(M+N)K_x \cdot \left|\frac{1}{\lambda^{\alpha}}\right|}{|(a*a)(t)|^{\alpha}}$$

$$\leq \frac{MK_x}{\epsilon_a^{\alpha}} \cdot \frac{1}{\lambda^{\alpha}\,|\,\hat{a}(\lambda)\,|} \cdot ((1*|a|)(t))^{1-\alpha} + \frac{(M+N)K_x}{\epsilon_a^{\alpha}} \cdot \left|\frac{1}{\lambda^{\alpha}}\right| \cdot ((1*|a|)(t))^{-\alpha}$$

$$\leq \frac{MK_x}{\epsilon_a^{\alpha}} \cdot \frac{1}{|\,\lambda\hat{a}(\lambda)\,|} \cdot \lambda^{1-\alpha}\,((1*|a|)(t))^{1-\alpha} + \frac{(M+N)K_x}{\epsilon_a^{\alpha}} \cdot \lambda^{-\alpha}\,((1*|a|)(t))^{-\alpha}.$$

The third inequality is realized under: $1 \leq a(t) \leq N$ for $t \geq 0$ and Assumption A1 with $p = 1 : |(a*a)(t)| \geq \epsilon_a(|a|*|a|)(t)$ and that $\left|\frac{1}{\lambda\hat{a}(\lambda)}\right| \leq 1.$

Substituting $\lambda_t = \frac{N_{\omega}}{(1*|a|)(t)} > \omega$ for $t \in]0, 1](\lambda_t \to \infty \text{ as } t \to 0)$ with $N_{\omega} = 1 + \omega(1*|a|)(1)$, we obtain

$$\frac{\|R(t)x - a(t)x\|}{|(a*a)(t)|^{\alpha}} \leq \frac{MK_x N_{\omega}^{1-\alpha}}{\epsilon_a^{\alpha}} + \frac{(M+N)K_x N_{\omega}^{-\alpha}}{\epsilon_a^{\alpha}},$$

for all $0 < t \leq 1$. Thus $\sup\limits_{0<t\leq 1} \frac{\|R(t)x - a(t)x\|}{|(a*a)(t)|^{\alpha}} < \infty$, and hence $x \in \widetilde{F}^{\alpha}(A)$.

(ii) Let $x \in \widetilde{F}^{\alpha}(A)$ be given, then $\sup\limits_{0<t\leq 1} \frac{\|R(t)x - a(t)x\|}{|(a*a)(t)|^{\alpha}} := J_x < \infty$. For $\lambda > \omega$, we write

$$AK(\lambda)x = \frac{1}{\hat{a}(\lambda)} K(\lambda)x - x,$$

then

$$\lambda^{\alpha} AK(\lambda)\, x = \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} K(\lambda)\, x - \lambda^{\alpha} x$$

$$= \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} [\, K(\lambda)\, x - \hat{a}(\lambda)\, x \,]$$

$$= \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{\infty} e^{-\lambda s} \, (R(s)\, x - a(s)\, x)\, ds$$

$$= \frac{\lambda^{\alpha}}{\hat{a}(\lambda)} \int_0^{\infty} e^{-\lambda s} . ((a * a)(s))^{\alpha} \, . \frac{R(s)\, x - a(s)\, x}{((a * a)(s))^{\alpha}}\, ds.$$

The fact that a is satisfying Assumption A2, gives us

$$\|\lambda^{\alpha} AK(\lambda) x\| \leq \frac{(L_{\alpha}\|x\| + J_x)}{\epsilon_{a,\alpha}} \text{ with } L_{\alpha} = \frac{M + N}{(1 * a)^{\alpha}(1)}.$$

Therefore, $\sup\limits_{\lambda > \omega} \|\lambda^{\alpha} AK(\lambda) x\| < \infty,$ which ends the proof.

□

Example 3.8. Let $\alpha \in \,]0, 1]$.

(i) Let $a(t) = 1$. Then a satisfies Assumption A1 with $p = 1$. Furthermore a satisfies Assumption A2 (see Example 3.6 (i)) and by virtue of Proposition 3.7 we obtain $F^{\alpha}(A) = \widetilde{F}^{\alpha}(A)$. Hence we recover a result for C_0-semigroups case which corresponds to [10, Proposition 5.12].

(ii) Let $a(t) = \frac{1}{\sqrt{t}}$ and $\alpha = \frac{1}{3}$. Then a satisfies Assumption A1 with $p = 1$. Furthermore a satisfies Assumption A2 (see Example 3.6 (ii)) and by virtue of Proposition 3.7 we obtain $F^{\frac{1}{3}}(A) = \widetilde{F}^{\frac{1}{3}}(A)$.

Remark 3.9. We notice that we have not used the case $p > 1$ in the Assumption A1, we will use it in a future work on the notion of admissibility.

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