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STANLEY COHEN-MACAULAY MODULES

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Abstract

Let $S = K[x_1, \ldots, x_n]$ and M be a finitely generated $\mathbb{Z}^n$-graded S-module. We say that M is a Stanley Cohen-Macaulay module if $\operatorname{sdim}(M) = \operatorname{sdepth}(M)$, where $\operatorname{sdim}(M) = \min\{\dim(S/P) : P \in \operatorname{Ass}(M)\}$. Let Δ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$ with $\dim \Delta < n - 2$. Let F be an arbitrary face of $\Delta^{\vee}$ and x_0 be a new vertex. A cone from x_0 over F, denoted by $co_{x_0}F$, is the simplex on the vertex set $F \cup \{x_0\}$. Set $\Gamma = \Delta^{\vee} \cup co_{x_0}F$ and $\Delta' = \Gamma^{\vee}$. It is shown that if $K[\Delta']$ is a Stanley Cohen-Macaulay module then Stanley's conjecture holds for $K[\Delta]$. Moreover, we show that for a monomial ideal I of S if $x_t \in S$ is a regular element on S/I for some $1 \leq t \leq n$, then I is a Stanley Cohen-Macaulay ideal if and only if (I, x_t) is a Stanley Cohen-Macaulay ideal.

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Introduction

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K and M be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $m \in M$ be a homogeneous element in M and $Z \subseteq \{x_1, \dots, x_n\}$. We denote by $mK[Z]$ the K-subspace of M generated by all elements mf, where f is a monomial in $K[Z]$. The $\mathbb{Z}^n$-graded K-subspace $mK[Z] \subset M$ is called a Stanley space of dimension $|Z|$, if $mK[Z]$ is a free $K[Z]$-module. A Stanley decomposition of M is a presentation of the K-vector space M as a finite direct sum of Stanley spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i].$$

Set $\operatorname{sdepth}(\mathcal{D}) = \min\{[Z_i] : i = 1, \dots, r\}$. The number

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called Stanley depth of M.

Stanley [5] conjectured that $\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$ for all finitely generated $\mathbb{Z}^n$-graded S-modules M.

Let M be a finitely generated $\mathbb{Z}^n$-graded S-module.

A chain of $\mathbb{Z}^n$-graded submodules $\mathcal{F} : 0 = M_0 \subset M_1 \subset \dots \subset M_r = M$ is called a prime filtration of M if $M_i / M_{i-1} \cong S / P_i(-a_i)$, where $a_i \in \mathbb{Z}^n$ and each P_i is a monomial prime ideal. We call the set $\{P_1, \dots, P_r\}$ the support of $\mathcal{F}$ and denote it $\operatorname{supp}(\mathcal{F})$.

Herzog et al. proved in [2, Proposition 1.3] that if $\mathcal{F}$ is a prime filtration of M, then

$\min\{\dim(S / P) : P \in \mathcal{F} \leq \operatorname{depth}(M), \operatorname{sdepth}(M) \leq \min\{\dim(S / P) : P \in \operatorname{Ass}(M)\}$.

In [1, Proposition 1.2.13], it was shown that $\operatorname{depth}(M) \leq \min\{(S / P) : P \in \operatorname{Ass}(M)\}$.

In this paper, we study some results about Stanley Cohen-Macaulay S-modules. For this, we say that a finitely generated $\mathbb{Z}^n$-graded S-module M is Stanley Cohen-Macaulay module if sdim(M) = sdepth(M). We denote $\operatorname{sdim}(M) = \min\{\dim(S / P) : P \in \operatorname{Ass}(M)\}$.

This paper is organized as follows. In Section 1, we recall some notation and definitions which will be needed later. In Section 2, we study some results about Stanley Cohen-Macaulay ideals and Stanley dimensions. For this, we say that a monomial ideal I is Stanley Cohen-Macaulay ideal if S / I is a Stanley Cohen-Macaulay S-module. As a main result of this section we prove that for a monomial ideal I of S if $x_t \in S$ is a regular element on S / I for some $1 \leq t \leq n$. Then I is a Stanley Cohen-Macaulay ideal if and only if (I, x_t) is a Stanley Cohen-Macaulay ideal, see Theorem 2.4. Also, in the general case, we show that if M is a Stanley Cohen-Macaulay module, then M is not Cohen-Macaulay module, see Example 2.1. In Section 3, we let Δ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$ with $\dim \Delta < n - 2$. It is shown that if $K[\Delta']$ is a Stanley Cohen-Macaulay module, then Stanley's conjecture holds for $K[\Delta]$, where $\Delta' = \Gamma^\vee$, $\Gamma = \Delta^\vee \cup co_{x_0}F$ and $co_{x_0}F$ is the simplex on the vertex set $F \cup \{x_0\}$, see Theorem 3.2.

1. Preliminaries

In this section we fix some notation and recall some definitions.

A *simplicial complex* Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V, with the property that:

(a) $\{x_i\} \in \Delta$, for all i;

(b) if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set).

An element of Δ is called a *face* of Δ and complement of a face F is $V \setminus F$ and it is denoted by F^c. Also, the complement of the simplicial complex $\Delta = \langle F_1, \ldots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \ldots, F_r^c \rangle$. The *dimension* of a face F of Δ, $\dim F$, is $|F| - 1$, where $|F|$ is the number of elements of F and $\dim \varnothing = -1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of Δ is a subset F of V with $F \notin \Delta$, we denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of Δ. The maximal faces of Δ under inclusion are called *facets* of Δ. The *dimension* of the simplicial complex Δ, $\dim \Delta$, is the maximum of dimensions of its facets. If all facets of Δ have the same dimension, then Δ is called *pure*. For $F \subset \{x_1, \ldots, x_n\}$, we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S / I_\Delta$ the *Stanley-Reisner ring* of Δ. We define $\Delta^\vee$, the *Alexander dual* of Δ, by

$$\Delta^\vee = \{V \setminus F : F \notin \Delta\}.$$

It is known that for the complex Δ one has $I_{\Delta^\vee} = I(\Delta^c)$. For a square-free monomial ideal $I = (M_1, \dots, M_q) \subset S = K[x_1, \dots, x_n]$, the *Alexander dual* of *I*, denoted by $I^\vee$, is defined to be:

$$I^\vee = P_{M_1} \cap \cdots \cap P_{M_q},$$

where P_{M_i} is prime ideal generated by $\{x_j : x_j | M_i\}$.

Definition 1.1. Let *M* be a finitely generated $\mathbb{Z}^n$-graded S-module. Then the Stanley dimension of *M* is given by

$$\text{sdim}(M) = \min\{\dim(S/P) : P \in \text{Ass}(M)\}.$$

Definition 1.2. Let *M* be a finitely generated $\mathbb{Z}^n$-graded S-module. *M* is Stanley Cohen-Macaulay module if $\text{sdim}(M) = \text{sdepth}(M)$.

We also say that $x \in S$ is an *M*-regular element if $xz = 0$ for $z \in M$ implies $z = 0$, in other words, if *x* is not a zero-divisor on *M*.

Remark 1.3. Let *M* be a finitely generated $\mathbb{Z}^n$-graded S-module, and M_P be localization of *M* with respect to prime ideal *P*. Then it is easy to see that $\text{sdim}(M_P) = 0$.

Example 1.4. Let $I = (\{x_i x_j : 1 \leq i < j \leq m\})$ be a monomial ideal of *S*. Villarreal [7] showed that this ideal is the edge ideal of a complete graph. On the other hand, complete graphs are Cohen-Macaulay so *I* is a Cohen-Macaulay ideal. Therefore S / I is a Cohen-Macaulay. $\text{Ass}(S/I) = \{(x_1, \dots, \widehat{x_i}, \dots, x_n)\}$, where $(x_1, \dots, \widehat{x_i}, \dots, x_n)$ means that omit variable x_i.

Without loss of generality consider $P = (x_1, \dots, x_{n-1})$.

Now localize (S / I) with respect to prime ideal P so $(S / I)_P = (S_P / PS_P)$. By part (i) of Lemma 2.2 one has $\operatorname{sdim}(S / I) = \dim(S / I) = 1$ and $\operatorname{sdim}(S / I)_P = \operatorname{sdim}(S_P / PS_P) = \dim(S_P / PS_P) = 0$.

Definition 1.5. A monomial ideal I is called Stanley Cohen-Macaulay ideal if S / I is a Stanley Cohen-Macaulay S-module.

2. Stanley Cohen-Macaulay Ideals and Stanley Dimensions

Let $I \subset S$ be a monomial ideal of $S = K[x_1, \dots, x_n]$ and $x_t \in S$ for some $1 \leq t \leq n$ be regular on S / I. In this section, we show that I is a Stanley Cohen-Macaulay ideal if and only if (I, x_t) is a Stanley Cohen-Macaulay ideal. The following example shows that in general if M is a Stanley Cohen-Macaulay module, then M is not Cohen-Macaulay module.

Example 2.1. Let $S = K[x_1, x_2]$ and $M = S / (x_1^2, x_1x_2)$, then M is Stanley Cohen-Macaulay S-module. Because $\operatorname{sdim}(M) = \operatorname{sdepth}(M) = 0$. On the other hand, $\operatorname{depth}(M) = 0$, $\dim(M) = 1$.

Lemma 2.2. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module, and M Cohen-Macaulay module. Then*

(i) $\operatorname{sdim}(M) = \dim(M)$,

(ii) *If Stanley's conjecture holds for the module M, then M is a Stanley Cohen-Macaulay module.*

Proof. (i) Since $\operatorname{depth}(M) = \dim(S / P)$ for all $P \in \operatorname{Ass}(M)$ then $\operatorname{depth}(M) = \operatorname{sdim}(M)$.

On the other hand, $\operatorname{depth}(M) = \dim(M)$ therefore $\operatorname{sdim}(M) = \dim(M)$.

(ii) We know that $\operatorname{depth}(M) \leq \operatorname{sdepth}(M) \leq \operatorname{sdim}(M)$ and M is Cohen-Macaulay. So $\operatorname{sdim}(M) = \operatorname{sdepth}(M)$. □

Proposition 2.3. *Let* $I = (u_1, \dots, u_r)$ *be a monomial ideal, and* $x_t \in S$ *for some* $1 \le t \le n$ *be regular on* S / I. *Then*

(i) $x_t \nmid u_i$ *for all* $i = 1, \dots, r$;

(ii) $I = \bigcap_{i=1}^{l} Q_i$ *is the minimal primary decomposition of I if and only if* $(I, x_t) = \bigcap_{i=1}^{l} (Q_i, x_t)$ *is the minimal primary decomposition of* (I, x_t).

Proof. (i) Suppose on the contrary that there exists $d \in [r]$ such that $x_t \mid u_d$ and $u_d = x_t f_d$ for some $f_d \in S$.

This implies that $f_d \notin I$ and $x_t(f_d + I) = I$ which is a contradiction.

(ii) Let $I = \bigcap_{i=1}^{l} Q_i$ is the minimal primary decomposition of *I*. We claim that $\bigcap_{i=1}^{l} (Q_i, x_t)$ is the minimal primary decomposition of (I, x_t). We first prove that $(I, x_t) = \bigcap_{i=1}^{l} (Q_i, x_t)$. Let $u \in (I, x_t)$ then we have to consider two cases: If $x_t \nmid u$, then one has $u \in I$ and $u \in Q_i$ for all $i = 1, \dots, l$. This implies that $u \in (Q_i, x_t)$ for all $i = 1, \dots, l$ and $u \in \bigcap_{i=1}^{l} (Q_i, x_t)$. If $x_t \mid u$, then $u \in (Q_i, x_t)$ for all $i = 1, \dots, l$ so $u \in \bigcap_{i=1}^{l} (Q_i, x_t)$. Assume $w \in \bigcap_{i=1}^{l} (Q_i, x_t)$. Then we have $w \in (Q_i, x_t)$ for all $i = 1, \dots, l$. If $x_t \mid w$, then $w \in (I, x_t)$. Otherwise, we have $w \in Q_i$ for all $i = 1, \dots, l$ thus $w \in I$ and $w \in (I, x_t)$. Now we show that (Q_i, x_t) is a primary ideal for all $i = 1, \dots, l$. Let $fg \in (Q_i, x_t)$, where $f, g \in S$ and $g \notin (Q_i, x_t)$ and $g \notin Q_i$. If $fg \in Q_i$, then there exists $n \in \mathbb{N}$ such that $f^n \in Q_i$ so $f^n \in (Q_i, x_t)$.

If $fg \in (x_t)$ and $g \notin (x_t)$, then there exists $m \in \mathbb{N}$ such that $f^m \in (x_t)$ and $f^m \in (Q_i, x_t)$. It suffices to show that decomposition is minimal. Suppose on the contrary that there exists $d \in [l]$ such that $\bigcap_{i=1, i \neq d}^{l} (Q_i, x_t) \subset (Q_d, x_t)$ so $\bigcap_{i=1, i \neq d}^{l} Q_i \subset Q_d$, which is a contradiction. Let $(I, x_t) = \bigcap_{i=1}^{l} (Q_i, x_t)$ is the minimal primary decomposition of (I, x_t), then $(I, x_t) \setminus (x_t) = \bigcap_{i=1}^{l} ((Q_i, x_t) \setminus (x_t))$. Therefore $I = \bigcap_{i=1}^{l} Q_i$ is the minimal primary decomposition of I. □

Theorem 2.4. *Let* $I \subset S$ *be a monomial ideal of* $S = K[x_1, \dots, x_n]$ *and* $x_t \in S$ *for some* $1 \leq t \leq n$ *be regular on* S / I. *Then* $\operatorname{sdim}(S / (I, x_t)) = \operatorname{sdim}(S / I) - 1$. *In particular, I is a Stanley Cohen-Macaulay ideal if and only if* (I, x_t) *is a Stanley Cohen-Macaulay ideal.*

Proof. By part (ii) of Lemma 2.2 and the definition of Stanley dimension we have $\operatorname{sdim}(S / (I, x_t)) = \operatorname{sdim}(S / I) - 1$. Let I be a Stanley Cohen-Macaulay ideal, then $\operatorname{sdim}(S / I) = \operatorname{sdepth}(S / I)$. Also $\operatorname{sdim}(S / (I, x_t)) = \operatorname{sdim}(S / I) - 1$. On the other hand, Rauf [6, Theorem 2.4.1] proved that

$\operatorname{sdepth}(S / (I, x_t)) = \operatorname{sdepth}(S / I) - 1$. Therefore $\operatorname{sdim}(S / (I, x_t)) = \operatorname{sdepth}(S / (I, x_t))$. Now let (I, x_t) is a Stanley Cohen-Macaulay ideal, then

$$\operatorname{sdim}(S / (I, x_t)) = \operatorname{sdim}(S / I) - 1 = \operatorname{sdepth}(S / (I, x_t)) = \operatorname{sdepth}(S / I) - 1.$$

So $\operatorname{sdim}(S / I) = \operatorname{sdepth}(S / I)$. □

3. Simplicial Complexes and Stanley Cohen-Macaulay Modules

Let Δ be a simplicial complex on the vertex set $\{x_1, \dots, x_n\}$ with $\dim \Delta < n - 2$. Let F be an arbitrary face of $\Delta^\vee$ and x_0 be a new vertex. A cone from x_0 over F, denoted by $co_{x_0}F$, is the simplex on the vertex set $F \cup \{x_0\}$. Set $\Gamma = \Delta^\vee \cup co_{x_0}F$ and $\Delta' = \Gamma^\vee$. In this section we want to show that if $K[\Delta']$ is a Stanley Cohen-Macaulay module, then Stanley's conjecture holds for $K[\Delta]$. For the proof we shall need

Lemma 3.1. *Let Δ be a simplicial complex on the vertex set $\{x_1, \dots, x_n\}$ with $\dim \Delta < n - 2$. Let F be an arbitrary face of $\Delta^\vee$ and x_0 be a new vertex. Set $\Gamma = \Delta^\vee \cup co_{x_0}F$ and $\Delta' = \Gamma^\vee$. Let $S = K[x_1, \dots, x_n]$ and $\varphi : S[x_0] \to S$ be the K-algebra homomorphism with $x_i \mapsto x_i$ for $i = 1, \dots, n$ and $x_0 \mapsto 1$, then $\varphi(I_{\Delta'}) = I_\Delta$.*

Proof. Set $S = K[x_1, \dots, x_n]$, $S' = S[x_0]$ and $G(I_\Delta) = \{m_1, \dots, m_d\}$, where $G(I_\Delta)$ is the set of minimal monomial generators of I_Δ. Then

$$I_{\Delta^\vee} = P_{G_1} \cap \dots \cap P_{G_d} \subset S,$$

where $G_1, \dots, G_d$ are all facets of $\Delta^\vee$ and $m_j = \prod_{x_i \in G_j} x_i$. We may assume $F \subset G_1$ without loss of the generality. Then

$$I_\Gamma = P_{F \cup \{x_0\}} \cap (P_{G_1}S' + (x_0)) \cap \dots \cap (P_{G_d}S' + (x_0)) \subset S'.$$

Hence

$$I_{\Gamma^\vee} = I_{\Delta'} = \{m_0, x_0m_1, \dots, x_0m_d\}S',$$

where $m_0 = \prod_{x_i \in P_{F \cup \{x_0\}}} x_i$. Since $\{x_1, \dots, x_n\} \setminus G_1 \subset \{x_1, \dots, x_n\} \setminus F$, then m_0 is divisible by m_1. Now we prove that $\varphi(I_{\Delta'}) = I_\Delta$. Suppose

$u \in \varphi(I_{\Delta'})$, then there exists $v \in I_{\Delta'}$ such that $u = \varphi(v)$. If $x_0 \nmid v$, then $u = \varphi(v) = v$ and $u \in I_{\Delta}$. If $x_0 \mid v$, then we have to consider two cases.

Case 1. If $m_0 \mid v$, then $v = m_0 g$, where $g \in S'$. So $\varphi(v) = m_0\varphi(g)$ and $m_0 \mid u$ and $m_1 \mid u$. Therefore $u \in I_{\Delta}$.

Case 2. If $m_0 \nmid v$, then there exists $i \in [d]$ such that $x_0 m_i \mid v$ and $m_i \mid \varphi(v) = u$. Hence $\varphi(I_{\Delta'}) \subset I_{\Delta}$. We prove the opposite inclusion. We consider a monomial $w \in I_{\Delta}$. Then there exists $i \in [d]$ such that $m_i \mid w$ and $w = m_i f$, where $f \in S$. We set $w' = x_0 m_i f \in I_{\Delta'}$, then $\varphi(w') = w \in \varphi(I_{\Delta'})$ and $I_{\Delta} \subset \varphi(I_{\Delta'})$. □

Now we are ready to prove the main result of this paper.

Theorem 3.2. *Let* Δ *be a simplicial complex on the vertex set* $\{x_1, \ldots, x_n\}$ *with* $\dim \Delta < n - 2$. *Let F be an arbitrary face of* $\Delta^{\vee}$ *and* x_0 *be a new vertex. Set* $\Gamma = \Delta^{\vee} \cup co_{x_0} F$ *and* $\Delta' = \Gamma^{\vee}$. *If* $K[\Delta']$ *is a Stanley Cohen-Macaulay module, then Stanley's conjecture holds for* $K[\Delta]$.

Proof. Since $K[\Delta']$ is a Stanley Cohen-Macaulay module then $\operatorname{depth}(K[\Delta']) \leq \operatorname{sdepth}(K[\Delta'])$. By Lemma 3.1 and [4, Corollary 3.2], we have $\operatorname{sdepth}(K[\Delta']) \leq \operatorname{sdepth}(K[\Delta]) + 1$. Kimura [3, Lemma 2.2] proved that projective dimensions of $K[\Delta']$ and $K[\Delta]$ are equal. So it is easy to see that $\operatorname{depth}(K[\Delta']) = \operatorname{depth}(K[\Delta]) + 1$. Therefore the assertion is proved. □

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