Research and Communications in Mathematics and Mathematical Sciences
Vol. 9, Issue 1, 2017, Pages 37-50
ISSN 2319-6939
Published Online on November 29, 2017

http://jyotiacademicpress.org

SUBSEQUENCE CHARACTERIZAT ION OF UNIFORM STATISTICAL CONVERGENCE OF DOUBLE SEQUENCE

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Abstract

In this paper, it is shown that almost every, in terms of P, subsequence $S(x)$ of double sequence S is not uniformly statistically convergent to L if S converges to L uniformly statistically.

Almost every, in terms of measure P_A, subsequence $S(x)$ of double sequence S converges to L, in the Pringsheim's sense, if S converges to L uniformly statistically and divergently in the Pringsheim's sense. This is not true for P.

1. Introduction

The concept of the statistical convergence of a sequences of reals was introduced by Fast [13]. Furthermore, Gökhan et al. [16] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued function. Çakan and Altay [5] presented multi-

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2010 Mathematics Subject Classification: Primary 40B05; Secondary 40A35, 40G15.
Keywords and phrases: double sequences, uniform statistical convergence.
Received October 26, 2017; Revised November 18, 2017

dimensional analogues of the results presented by Fridy, Miller and Orhan [14, 15, 17]. Dündar and Atay [6-10] investigated the relation between I-convergence of double sequences. Now, we recall that the definitions of concepts of ideal convergence and basic concepts [1, 2, 11, 12, 18].

The sequence S_{ij} of real numbers converges to L in the Pringsheim's sense, if for $A\varepsilon > 0, \exists K > 0$ such that

$$|S_{ij} - L| \leq \varepsilon, \forall i, j \geq K.$$

We write $\lim\limits_{i, j \to \infty} S_{ij} = L.$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{nm} be the number of $(i, j) \in K$ such that $i \leq n, j \leq m$. If

$$d_2(K) = \lim_{n, m \to \infty} \frac{K_{nm}}{nm}$$

in the Pringsheim's sense. Then we say that K has double natural density. Let is sequence S_{ij} of real numbers and $\varepsilon > 0$. Let

$$A(\varepsilon) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\}.$$

The sequence $S = S_{ij}$ statistically converges to $L \in \mathbb{R}$ if $d_2(A(\varepsilon)) = 0$ for $\forall \varepsilon > 0$.

We write $st - \lim S_{ij} = L.$

Let is set $X \neq \emptyset$. A class I of subsets of X is said to be an ideal in X provided the following statements hold:

(i) $\emptyset \in I$;

(ii) $A, B \in I \Rightarrow A \cup B \in I$;

(iii) $A \in I, B \subset A \Rightarrow B \in I$.

I is nontrivial ideal if $X \notin I$. A nontrivial ideal I is called admissible if $\{x\} \in I$ for $\forall x \in X$.

In this paper, the focus is put on ideal $I_u \subset 2^{\mathbb{N}\times\mathbb{N}}$ defined by: subset A belongs to the I_u if

$$\lim_{p,q\to\infty} \frac{1}{pq} |\{i < p,\ j < q : (n + i,\ m + j) \in A\}| = 0$$

uniformly on $n, m \in \mathbb{N}$ in the Pringsheim's sense. That is subset A of the set $\mathbb{N} \times \mathbb{N}$ is uniformly statistically density zero.

The sequence $S = S_{ij}$ uniformly statistically converges to L if for any $\varepsilon > 0$

$$\{(i,\ j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\} \in I_u.$$

That is sequence $S = S_{ij}$ uniformly statistically converges to L, if $\forall\varepsilon, \varepsilon' > 0, \exists K > 0$ such that

$$\frac{1}{pq} |\{i < p,\ j < q : |S_{n+i,\, m+j} - L| \geq \varepsilon\}| < \varepsilon',\ \forall p,\ q \geq K,\ \forall n,\ m \in \mathbb{N}.$$

We write $Ust - \lim S_{ij} = L$.

We denote with X a set of all double sequences of 0's and 1's, i.e.,

$$X = \{x = x_{ij} : x_{ij} \in \{0, 1\},\ i,\ j \in \mathbb{N}\}.$$

Let sequence $S = S_{ij}$ and $x \in X$. Then with $S(x)$ we denote a sequence defined following way:

$$S_{ij}(x) = S_{ij},\ \text{for}\ x_{ij} = 1,$$

which we refer to as subsequence of sequence S.

The mapping $x \to S(x)$ is a bijection of the set X to a set of all subsequences of the sequence S.

Then, under the Lebesgue measure on the set of all subsequences of the sequence S consider Lebesgue measure on the set X.

Let β smallest σ-algebra subsets of the set X which contains of subsets in the form of:

$$\{x = (x_{nm}) \in X : x_{n_1m_1} = a_1, x_{n_2m_2} = a_2, \cdots, x_{n_km_k} = a_k\},$$

$$a_1, \cdots, a_k \in \{0, 1\}, k \in \mathbb{N}.$$

In [3], it was proven that there is a unique Lebesgue measure P on the set X for which the following applies:

$$P(\{x = (x_{nm}) \in X : x_{n_1m_1} = a_1, x_{n_2m_2} = a_2, \cdots, x_{n_km_k} = a_k\}) = \frac{1}{2^k}.$$

2. New Results

Almost every double sequences of 0's and 1's is not almost convergent [4]. This analogue is valid also for uniform statistical convergence.

Theorem 2.1. *Almost every double sequence of* 0*'s and* 1*'s is not uniformly statistically convergent.*

Proof. Let

$$A_n^k = \{x \in X : x_{ij} = 1, k \le i, j < k + n\}.$$

Since $P(A_n^k) = \frac{1}{2^{n^2}}, \forall k \in \mathbb{N}$, it is,

$$\sum_{k=1}^{\infty} P(A_n^k) = \sum_{k=1}^{\infty} \frac{1}{2^{n^2}} = +\infty.$$

Since A_n^k are independent, based on the second part of Borel-Cantelli lemma:

$$P\left(\limsup_k A_n^k\right) = 1.$$

We denote $A_n = \limsup_k A_n^k,\ A = \bigcap_{n=1}^{\infty} A_n$. Then, $P(A) = 1$. For $\forall x \in A$, $\forall n \in \mathbb{N}$, there exist a block $n \times n$ composed of ones. It follows $\forall x \in A$ does not converge uniformly statistically to 0. We denote

$$B_n^k = \{x \in X : x_{ij} = 0,\ k \le i,\ j < k + n\},\ B_n = \limsup_k B_n^k,\ B = \bigcap_{n=1}^{\infty} B_n.$$

Completely analogously, we conclude that $P(B) = 1$. For $\forall x \in B$, $\forall n \in \mathbb{N}$, there exists a block $n \times n$ composed of zeros. It follows $\forall x \in B$ does not converge uniformly statistically to 1.

Every uniformly statistically convergent sequence $x \in X$ converges to 0 or 1. Then,

$$\{x \in X : x = (x_{ij}) \text{ convergent uniformly statistically}\}$$

$$= \{x \in X : Ust - \lim x_{ij} = 0\} \bigcup \{x \in X : Ust - \lim x_{ij} = 1\}$$

$$= A^c \bigcup B^c = (A \bigcap B)^c.$$

It follows

$$P(\{x \in X : x = (x_{ij}) \text{ convergent uniformly statistically}\}) = 1 - P(A \bigcap B) = 0.$$

Definition 2.2. The subsequence $S(x)$ of sequence S uniformly statistically converges to L, if $\forall \varepsilon, \varepsilon' > 0, \exists K > 0$ such that for $\forall p, q \ge K$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{i < p,\ j < q : |S_{n+i, m+j} - L| \ge \varepsilon,\ x_{n+i, m+j} = 1\}|}{|\{i < p,\ j < q : x_{n+i, m+j} = 1\}|} \le \varepsilon'.$$

We write $Ust - \lim S_{ij}(x) = L$.

Almost every subsequence $S(x)$ of statistically convergent double sequence S is statistically convergent. The analogue does not apply to uniformly statistically convergence.

Theorem 2.3. *Let* $Ust - \lim S_{ij} = L$ *and the sequence* S *is divergent in the Pringsheim's sense. Then,*

$$P(\{x \in X : Ust - \lim S_{ij}(x) = L\}) = 0.$$

Proof. Let

$$T_{uv}^{k} = \Big\{x \in X : x_{nk+i,\, nk+j} = 1,\ |S_{nk+i,\, nk+j} - L| \geq \varepsilon,$$

$$0 < i,\ j < k,\ n = k\frac{u(u+1)}{2},\ m = k\frac{v(v+1)}{2}\Big\}.$$

Due to the divergence of the sequence S, $\forall N \in \mathbb{N}$, $\exists k \geq N$, such that $T_{uv}^{k} \neq \emptyset$ for infinitely (u, v). Then, $P(T_{uv}^{k}) \geq \frac{1}{2^{k^2}}$ for infinitely (u, v), T_{uv}^{k} are independent and

$$\sum_{(u,v),\, T_{uv}^{k} \neq \emptyset} P(T_{uv}^{k}) \geq \sum_{(u,v),\, T_{uv}^{k} \neq \emptyset} \frac{1}{2^{k^2}} = +\infty.$$

Due to second part of Borel-Cantelli lemma, it follows:

$$P(T^{k}) = P\left(\limsup_{(u,v),\, T_{uv}^{k} \neq \emptyset} T_{uv}^{k}\right) = 1.$$

Hence, $P(T) = P\left(\bigcap_{k} T^{k}\right) = 1.$

Let $x \in T$. Then, $\forall N \in \mathbb{N}$, $\exists k \geq N$ such that

$$\frac{|\{x \in X : x_{nk+i,\, nk+j} = 1, |S_{nk+i,\, nk+j} - L| \geq \varepsilon,\ 0 < i,\ j < k\}|}{|\{i < p,\ j < q : x_{n+i,\, m+j} = 1\}|} = 1.$$

Hence, $S(x)$ does not converge to L uniformly statistically. So,

$$P(\{x \in X : Ust - \lim S_{ij}(x) = L\}) = 0.$$

Example. Let $A \subset \mathbb{N} \times \mathbb{N}$ uniformly density zero, with the following characteristic:

$$\forall k \in \mathbb{N}, \exists (i, j) \in A, \text{ such that } i, j \geq k.$$

Let the sequence $S = (S_{ij})$ defined as

$$S_{ij} = \begin{cases} 1, & (i, j) \notin A \\ 0, & (i, j) \in A \end{cases}.$$

Then, $\forall \varepsilon > 0, \forall n, m \in \mathbb{N}$, the following is valid:

$$\frac{1}{pq} |\{i < p, j < q : |S_{n+i, m+j} - 1| \geq \varepsilon\}|$$

$$= \frac{1}{pq} |\{i < p, j < q : (n + i, m + j) \in A\}| \to 0 \text{ for } p, q \to \infty.$$

Respectively, $Ust - \lim S_{ij} = 1$. Let

$$B = \bigcap_{k=1}^{\infty} \bigcup_{i, j \geq k}^{\infty} \{x \in X : x_{ij} = 1, (i, j) \in A\}.$$

Let is the infinite sequence $(S_{i_k j_k})$ such that $i_k \geq k, j_k \geq k$ and $(i_k, j_k) \in A$. Then,

$$\sum_{k=1}^{\infty} P(\{x \in X : x_{i_k j_k} = 1\}) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty.$$

Due to second part of Borel-Cantelli lemma, $P(B) = 1$.

Since subsequence $S(x)$ of S does not converge to 1 in the Pringsheim's sense if and only if $x \in B$, it

$$P(\{x \in X : \lim_{i, j \to \infty} S_{ij}(x) = 1, \text{ in the Pringsheim's sense}\}) = 0.$$

Let $A \subset \mathbb{N} \times \mathbb{N}$. There is a unique measure P_A on X with the property:

$$P_A(\{x \in X : x_{ij} = 1\}) = \begin{cases} \frac{1}{2}, & (i, j) \notin A \\ \frac{1}{2^{i+j}}, & (i, j) \in A \end{cases},$$

$$P_A(\{x \in X : x_{i_1 j_1} = a_1, \cdots, x_{i_k j_k} = a_k\})$$

$$= P_A(\{x \in X : x_{i_1 j_1} = a_1\}) \cdots P_A(\{x \in X : x_{i_k j_k} = a_k\}).$$

Analogue theorem is valid: Let the sequence $S = (S_{ij})$ be divergent in the Pringsheim's sense. Then, S statistically converges to $L \Leftrightarrow \exists A \subset \mathbb{N} \times \mathbb{N}$ with density zero, such that

$$P_A(\{x \in X : \lim_{i, j \to \infty} S_{ij}(x) = L, \text{ in the Pringsheim's sense}\}) = 1.$$

Theorem 2.4. *Let the sequence* $S = (S_{ij})$ *divergent in the Pringsheim's sense. Then,* S *uniformly statistically converges to* $L \Leftrightarrow \exists A \subset \mathbb{N} \times \mathbb{N}$ *uniformly density zero, such that*

$$P_A(\{x \in X : \lim_{i, j \to \infty} S_{ij}(x) = L, \textit{ in the Pringsheim's sense}\}) = 1.$$

Proof. Because of Lemma 2.1, the following is valid: $Ust - \lim S_{ij} = L \Rightarrow \exists A \subset \mathbb{N} \times \mathbb{N}$ uniformly density zero, such that the subsequence $S(y)$ of S converges to L, in the Pringsheim's sense, for

$$y_{ij} = \begin{cases} 1, & (i, j) \notin A \\ 0, & (i, j) \in A \end{cases}.$$

Not generalizing we can assume that L is not a point accumulation of the subsequence $S(x)$, for

$$x_{ij} = \begin{cases} 1, & (i, j) \in A \\ 0, & (i, j) \notin A \end{cases}.$$

Hence, the subsequence $S(z)$ converges to L, in the Pringsheim's sense if and only if $\exists M \in \mathbb{N}$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : z_{ij} = 1, i, j \geq M\} \bigcap A = \emptyset.$$

Let

$$B_M = \{x \in X : x_{ij} = 1, i, j \geq M, (i, j) \in A\}, B = \bigcap_{M=1}^{\infty} B_M.$$

Then, $\forall M \in \mathbb{N}$, is,

$$P_A(B) \leq P_A(B_M) = \sum_{i, j \geq M, (i, j) \in A} \frac{1}{2^{i+j}} \leq \sum_{i, j \geq M} \frac{1}{2^{i+j}} = \frac{1}{2^{2M-2}}.$$

Hence, $P_A(B) = 0$. Since the set B is a set of all $x \in X$ for which $S(x)$ does not converge to L, in the Pringsheim's sense. It

$$P_A(\{x \in X : \lim_{i, j \to \infty} S_{ij}(x) = L, \text{ in the Pringsheim's sense}\}) = 1.$$

Let the sequence S not be uniformly statistically convergent and let $A \subset \mathbb{N} \times \mathbb{N}$ arbitrary uniformly density zero. Then, due to the lemma, the subsequence $S(x)$ is divergent in the Pringsheim's sense for

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A \\ 0, & (i, j) \in A \end{cases}.$$

The following cases can be presented:

(a) $\exists (n_k), (m_k), n_k \nearrow, m_k \nearrow, (n_k, m_k) \notin A, S_{n_k m_k} \geq k$ for $\forall k$,

(b) $\exists (n_k), (m_k), n_k \nearrow, m_k \nearrow, (n_k, m_k) \notin A, S_{n_k m_k} \leq -k$ for $\forall k$,

(c) $\exists (n_k^1), (m_k^1), \exists (n_k^2), (m_k^2), (n_k^1, m_k^1), (n_k^2, m_k^2) \notin A, S_{n_k^1 m_k^1} \leq \lambda$

$< \mu \leq S_{n_k^2 m_k^2}$.

It follows:

(a) $\sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k m_k} = 1\}) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty,$

(b) $\sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k m_k} = 1\}) = \sum_{k=1}^{\infty} \frac{1}{2} = \infty,$

(c) $\sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k^1 m_k^1} = 1\}) = \sum_{k=1}^{\infty} P_A(\{x \in X : x_{n_k^2 m_k^2} = 1\}) = \infty.$

Then, due to second part of Borel-Cantelli lemma, the following is valid:

(a) $P_A(\{x \in X : x_{n_k m_k} = 1 \text{ for infinite } k\}) = 1,$

(b) $P_A(\{x \in X : x_{n_k m_k} = 1 \text{ for infinite } k\}) = 1,$

(c) $P_A(\{x \in X : x_{n_k^1 m_k^1} = x_{n_k^2 m_k^2} = 1 \text{ for infinite } k\}) = 1.$

It follows:

$$P_A(\{x \in X : S(x) \text{ divergent in the Pringsheim's sense}\}) = 1.$$

Hence,

$$P_A(\{x \in X : S(x) \text{ convergent in the Pringsheim's sense}\}) = 0.$$

Lemma 2.5. $Ust - \lim S_{ij} = L \Leftrightarrow \exists A \subset \mathbb{N} \times \mathbb{N}$ *uniformly density zero such that* $\lim_{i,j\to\infty} S_{ij}(x) = L,$ *in the Pringsheim's sense for*

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A \\ 0, & (i, j) \in A \end{cases}.$$

Proof. Let $Ust - \lim S_{ij} = L.$ Then, there is a sequence of natural numbers $(r_k)_{k=2}^{\infty}$ such that

$$\frac{1}{pq}\left|\left\{i < p,\ j < q : |S_{n+i,\, m+j} - L| \geq \frac{1}{k}\right\}\right| \leq \frac{1}{k^2},\ \forall p,\ q \geq r_k,\ \forall n,\ m \in \mathbb{N}.$$

Let

$$A = \bigcup_{k=2}^{\infty} \bigcup_{n, m=1}^{\infty}$$

$$\left\{(n + i,\ m + j) : i,\ j \geq r_k,\ i < r_{k+1} \vee j < r_{k+1},\ |S_{n+i, m+j} - L| \geq \frac{1}{k}\right\}.$$

We define $x \in X$ the following way:

$$x_{ij} = \begin{cases} 1, & (i,\ j) \notin A \\ 0, & (i,\ j) \in A \end{cases}.$$

For $\forall \varepsilon > 0,\ \exists k_0 \in \mathbb{N},$ such that $\frac{1}{k} \leq \varepsilon,\ \forall k \geq k_0.$ From the definition of the sequence x, we have

$$|S_{n+i, m+j}(x) - L| = |S_{n+i, m+j} - L| \leq \varepsilon,\ \forall i,\ j \geq r_{k_0},\ \forall n,\ m \in \mathbb{N}.$$

Hence, the subsequence $S(x)$ converges to L in the Pringsheim's sense. For $\forall \varepsilon > 0,\ \exists k_0 \in \mathbb{N},$ such that,

$$\sum_{k=k_0}^{\infty} \frac{1}{k^2} \leq \frac{\varepsilon}{2},\ \frac{1}{(k_0 - 1)^2} \leq \frac{\varepsilon}{2}.$$

Let $p,\ q > r_{k_0}$, then

$$\frac{1}{pq} |\{i < p,\ j < q : (n + i,\ m + j) \in A\}|$$

$$\leq \frac{1}{pq} |\{i < p,\ j < q : i \leq r_{k_0} \vee j \leq r_{k_0},\ (n + i,\ m + j) \in A\}|$$

$$+ \frac{1}{pq} |\{i < p,\ j < q : i,\ j > r_{k_0},\ (n + i,\ m + j) \in A\}|.$$

$$\frac{1}{pq} |\{i < p,\ j < q : i \leq r_{k_0} \vee j \leq r_{k_0},\ (n + i,\ m + j) \in A\}|$$

$$\leq \frac{1}{pq} \left|\left\{i < p,\ j < q : |S_{n+i, m+j} - L| \geq \frac{1}{k_0 - 1}\right\}\right|$$

$$\leq \frac{1}{(k_0 - 1)^2} \leq \frac{\varepsilon}{2}, \ \forall n, m \in \mathbb{N}.$$

$$\frac{1}{pq} |\{i < p, j < q : i, j > r_{k_0}, (n+i, m+j) \in A\}|$$

$$\leq \frac{1}{pq} \left| \left\{ i < p, j < q : i, j > r_{k_0}, i \leq r_{k_0+1} \vee j \leq r_{k_0+1}, |S_{n+i,m+j} - L| \geq \frac{1}{k_0} \right\} \right|$$

$$+ \frac{1}{pq} \left| \left\{ i < p, j < q : i, j > r_{k_0+1}, i \leq r_{k_0+2} \vee j \leq r_{k_0+2}, |S_{n+i,m+j} - L| \geq \frac{1}{k_0+1} \right\} \right|$$

$$+ \cdots \leq \sum_{k=k_0}^{\infty} \frac{1}{k^2} \leq \frac{\varepsilon}{2}, \ \forall n, m \in \mathbb{N}.$$

Hence, $\forall \varepsilon > 0, \exists r_{k_0}$, such that

$$\frac{1}{pq} |\{i < p, j < q : (n+i, m+j) \in A\}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall p, q \geq r_{k_0}, \forall n, m \in \mathbb{N}.$$

Respectively, A is uniformly density zero.

We assume that there is a subset A of set $\mathbb{N} \times \mathbb{N}$, uniformly density zero such that subsequence $S(x)$ of S converges to L, in the Pringsheim's sense, for

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A \\ 0, & (i, j) \in A \end{cases}.$$

Then, $\forall \varepsilon > 0, \exists n_0, m_0 \in \mathbb{N}$, such that $|S_{ij} - L| \leq \varepsilon, \forall i \geq n_0, \forall j \geq m_0$.

$$\frac{1}{pq} |\{i < p, j < q : |S_{n+i,m+j} - L| \geq \varepsilon\}|$$

$$= \frac{1}{pq} |\{i < p, j < q : n+i < n_0 \vee m+j < m_0, |S_{n+i,m+j} - L| \geq \varepsilon\}|$$

$$+ \frac{1}{pq} |\{i < p, j < q : n+i \geq n_0, m+j \geq m_0, |S_{n+i,m+j} - L| \geq \varepsilon\}|$$

$$\leq \frac{1}{pq}\left|\{i < p,\ j < q : n + i < n_0 \vee m + j < m_0\}\right|$$

$$+ \frac{1}{pq}\left|\{i < p,\ j < q : n + i \geq n_0,\ m + j \geq m_0,\ (n + i,\ m + j) \in A\}\right|$$

$$\leq \frac{1}{pq}\left|\{i < p,\ j < q : n + i < n_0 \vee m + j < m_0\}\right|$$

$$+ \frac{1}{pq}\left|\{i < p,\ j < q : (n + i,\ m + j) \in A\}\right|.$$

Obviously, the first summand converges to zero uniformly on $n, m \in \mathbb{N}$. The second summand converges to zero uniformly on $n, m \in \mathbb{N}$ due to the assumption. So $Ust - \lim S_{ij} = L$.

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