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# TOWARD A MATHEMATICAL THEORY OF DYNAMICAL BEHAVIOUR IN COMPETITIVE MULTI-AGENT SYSTEMS

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## Abstract

A particularly important problem which awaits a full mathematical theory, concerns the dynamical behaviour of complex systems comprising multiple, interacting, heterogeneous components. There are many technological examples of such systems, including the so-called Internet-of-Things featuring collections of heterogeneous, autonomous machines and algorithms. Nature also arguably features examples in living systems, perhaps even the brain itself. The question therefore arises: How can such systems be represented mathematically and what features of their behaviour can be calculated? Here we discuss such a mathematical theory for representing the collective behaviour of a generic multi-agent population. A key feature of this approach is to account for the strong correlations that develop between agents' actions – and in particular, their strategies.

## 1. Introduction

It is now well-known that much of the natural, informational, sociological, and economic world features a complex system in one form of

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another [1-8]. Though there is no unique definition of a complex system, the qualification for classification can be thought of in terms of common features which distinguish it: specifically, many interacting components, non-stationarity, feedback and adaptation at the macroscopic and/or microscopic level, evolution, coupling with the environment, and dynamics that depend upon the particular realization of the system. Most importantly, complex systems have the ability to produce large macroscopic changes which appear spontaneously but have long-lasting consequences [9, 10].

John Casti has pointed out (see p. 213 of [4]) that '.... a decent mathematical formalism to describe and analyze the [so-called] El Farol Problem would go a long way toward the creation of a viable theory of complex, adaptive systems'. This El Farol Problem was originally proposed by Brian Arthur [11] to elucidate the key features of a complex system in a minimal yet generic everyday setting. The El Farol Problem features a collection of decision-making potential bar-goers, who repeatedly try to predict whether they should attend a potentially overcrowded bar on a given night each week. Though they have no information about the others predictions, they are each aware of the same string of previous outcomes ('overcrowded' or 'undercrowded') covering some limited number of previous occasions. As a result, no 'typical' agent exists, since all such typical agents would then make the same decision, hence rendering their common prediction scheme useless [12]. A simplified binary form of the El Farol Problem was introduced by Challet and Zhang [13] called the Minority Game (MG) [14-42]. The Minority Game features a population of $N$ heterogeneous agents with limited capabilities and information, who repeatedly compete to be in the minority group [43, 44].

Here we give a presentation of a mathematical theory for the emergent phenomena observed in such a system. Specifically, we present a mathematical theory for the Minority Game [26, 27] which can explain the emergent ability of the system to exhibit large changes which are well beyond the fluctuations expected in a null model (i.e., coin-toss) system. When tossing $N$ coins, it is well-known that it becomes practically impossible that nearly all show 'Heads' as $N$ becomes large (i.e., vanishingly small probability). However instead of the usual resulting $\sqrt{N}$ behaviour from such a null model coin-toss (i.e., random) system, we show in this paper that the fluctuations can be of order $N$ and hence of order of the size of the system itself. This emergent property will obviously play a very important role in determining what could go wrong in a real-world system, and with what probability. Likewise, ignoring it will significantly undervalue the risk of large changes. The mathematical theory that we present to describe this is built around the idea of crowding (i.e., correlations) in strategy-space, rather than the precise rules of the game itself, and it only makes modest assumptions about the game's dynamical behaviour. Hence it should have rather general applicability to populations of semi-autonomous, adaptive objects being fed the same information.

The contents of the paper are as follows. In Section 2, we give a background to the problem. In Section 3, we introduce and specify the model. In Section 4, we present the Crowd-Anticrowd formalism that we use to mathematically describe the emergence of large changes in the system. In Section 5, we provide an application of this to a generalized version of the model, the alloy Minority Game. Finally, in Section 6, we provide our conclusions.

## 2. Background

The mathematical approach is to incorporate accurately the correlations in strategies followed by the agents. Specifically, the Crowd-Anticrowd theory that we adopt considers groups containing like-minded agents ('Crowd') and opposite-minded agents ('Anticrowd'), in such a way that the strong strategy correlations are confined within each group, leaving weakly interacting Crowd-Anticrowd groups which then behave in an uncorrelated way with respect to each other. Each group $G$ contains a crowd of agents using strategies which are positively-correlated, and a complementary anticrowd using strategies which are strongly negatively-correlated to the crowd.

Hence a given group $G$ might contain $n_R[t]$ agents who are all using strategy $R$ and hence act as a crowd (e.g., by attending the bar en masse in the El Farol Problem) together with $n_{\overline{R}}[t]$ agents who are all using the opposite strategy $\overline{R}$ and hence act as an anticrowd (e.g., by staying away from the bar en masse). Most importantly, the anticrowd $n_{\overline{R}}[t]$ will always take the *opposite* decisions to the crowd $n_R[t]$ *regardless* of the current circumstances in the game, since the strategies $R$ and $\overline{R}$ imply the opposite action in all situations. Since all the strong correlations have been accounted for within each group, these individual groups $\{G\} = G_1, G_2, \ldots, G_n$ will then act in an uncorrelated way with respect to each other, and hence can be treated as $n$ uncorrelated stochastic processes. The global dynamics of the system is then given by the sum of the $n$ uncorrelated stochastic processes generated by the groups $\{G\} = G_1, G_2, \ldots, G_n$. We note that alternative theories have been proposed [15-19, 21, 37, 38, 40, 41], however such theories are unable to reproduce the original numerical results of [23] over the full range of parameter space, since they miss an accurate description of the correlations between agents' strategies.

### 3. Setup of the B-A-R (Binary Agent Resource) System

The generic form of the B-A-R (Binary Agent Resource) system is shown in Figure 1. At timestep $t$, each agent (e.g., a bar customer, a commuter, or a market agent) decides whether to enter a game where the choices are action $+1$ (e.g., attend the bar, take route A, or buy) and action $-1$ (e.g., go home, take route B, or sell). We denote the number of agents choosing $-1$ as $n_{-1}[t]$, and the number choosing $+1$ as $n_{+1}[t]$. The 'excess demand' is

$$D[t] = n_{+1}[t] - n_{-1}[t]. \tag{1}$$

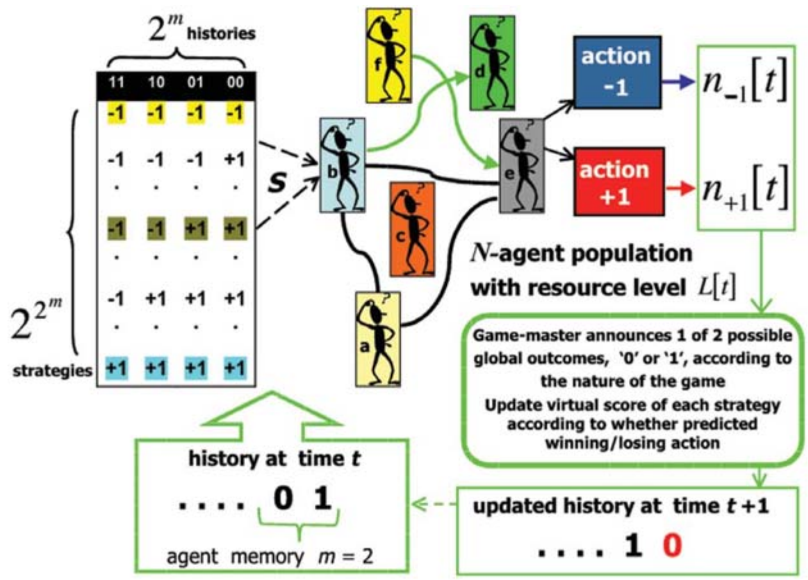


**Figure 1.** Schematic of B-A-R (Binary Agent Resource) system. At timestep $t$, each agent decides between action $-1$ and action $+1$ based on the predictions of the $S$ strategies that he possesses. A total of $n_{-1}[t]$ agents choose $-1$, and $n_{+1}[t]$ choose $+1$. In the simplified case that each agent's confidence threshold for entry into the game is very small, then $n_{-1}[t] + n_{+1}[t] = N$ (i.e., all agents play at every timestep). Agents may be subject to some underlying network structure which may be static or evolving, and ordered or disordered. The strategy allocation, and hence heterogeneity in the population, provides a further source of disorder which may be static (i.e., quenched) or evolving. The 'Game-master' aggregates the agents' actions and then announces the global outcome. All strategies are then rewarded/penalized according to whether they had predicted the winning/losing action. Adapted from [43].

The global information available to the agents is a common memory of the recent history, i.e., the most recent $m$ global outcomes. For example, for $m = 2$, the possible forms are $\ldots 00, \ldots 01, \ldots 10$ or $\ldots 11$ which we denote simply as $00, 01, 10$ or $11$. Hence at each timestep, the recent history constitutes a particular bit-string of length $m$. For general $m$, there will be $P = 2^m$ possible history bit-strings. These history bit-strings can alternatively be represented in decimal form: $\mu = \{0, 1, \dots, P-1\}$, where $\mu = 0$ corresponds to 00, $\mu = 1$ corresponds to 01 etc. A strategy consists of a predicted action, $-1$ or $+1$, for each possible history bit-string. Hence there are $2^P$ possible strategies. For $m = 2$, for example, there are therefore 16 possible strategies. In order to mimic the heterogeneity in the system, a typical game setup would have each agent randomly picking $S$ strategies at the outset of the game. In the Minority Game, these strategies are then fixed for all time – however a more general setup would allow the strategies held, and hence the heterogeneity in the population, to change with time. The agents then update the scores of their strategies after each timestep with $+1$ (or $-1$) as the pay-off for predicting the action which won (or lost). This binary representation of histories and strategies is due to Challet and Zhang [14]. The rules of the game determine the subsequent game dynamics, and can be generalized in principle to a range of real-world systems [43, 45, 46].

In the language of the El Farol Problem, we define the action $+1(-1)$ to be attend (stay away) with $L[t]$ representing the bar capacity. If $n_{+1}[t] < L[t]$, the bar will be under crowded and hence can be assigned the global outcome 0. Hence the winning action is $+1$ (i.e., attend). Likewise if $n_{+1}[t] > L[t]$, the bar will be overcrowded and can be assigned the global outcome 1. The winning action is then $-1$ (i.e., stay away). However more generally, the global outcome and hence winning action may be any function of present or past system data. Furthermore, the resource level $L[t]$ may be endogenously produced (e.g., a specific function of past values of $n_{-1}[t], n_{+1}[t]$) or exogenously produced (e.g. determined by external environmental concerns). In the basic game setup, we consider the agents to play their highest scoring strategy [35, 43, 47].

Figure 2 shows in more detail the $m = 2$ example strategy space from Figure 1. A strategy is a set of instructions to describe what an agent should do in any given situation, i.e., given any particular history $\mu$, the strategy then decides what action the agent should take. The strategy space is the set of strategies from which agents are allocated their strategies. If this strategy allocation is fixed randomly at the outset, then this acts as a source of quenched disorder. The strategy space shown is known as the Full Strategy Space (FSS), and contains all possible permutations of the actions $-1$ and $+1$ for each history. As such there are $2^{2^m}$ strategies in this space. The $2^m$ dimensional hypercube shows all $2^{2^m}$ strategies from the FSS at its vertices.

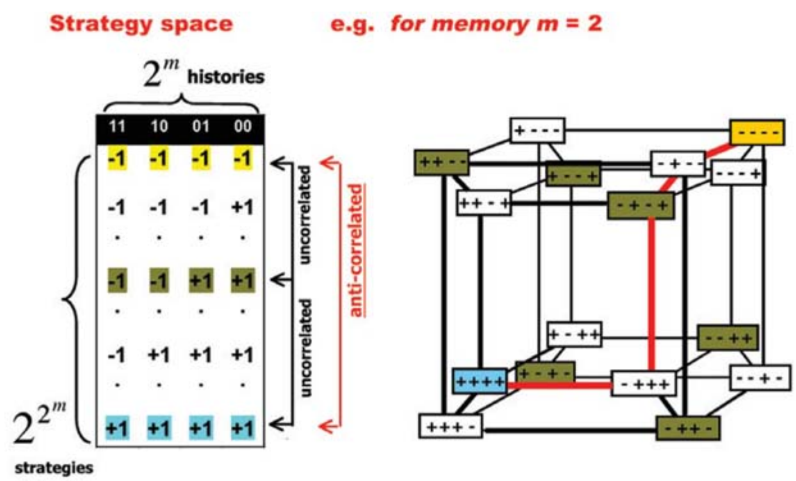


**Figure 2.** Strategy space for $m = 2$, together with some example strategies (left). The strategy space shown is known as the Full Strategy Space (FSS), and contains all possible permutations of the actions $-1$ and $+1$ for each history. There are $2^{2^m}$ strategies in the FSS. The $2^m$ dimensional hypercube (right) shows all $2^{2^m}$ strategies from the FSS at its vertices. The shaded strategies form a Reduced Strategy Space (RSS). There are $2.2^m = 2^{m+1}$ strategies in the RSS. The red shaded line connects two strategies with a Hamming distance separation of 4. Adapted from [43].

One can choose a subset of strategies [14] such that any pair within this subset has one of the following characteristics:

- anti-correlated, e.g., $-1-1-1-1$ and $+1+1+1+1$, or $-1-1+1+1$ and $+1+1-1-1$. For example, any two agents using the $(m=2)$ strategies $-1-1+1+1$ and $+1+1-1-1$, respectively, would take the opposite action irrespective of the sequence of previous outcomes and hence the history. Hence one agent will always do the opposite of the other agent. For example, if one agent chooses $+1$ at a given timestep, the other agent will choose $-1$. Their net effect on the demand $D[t]$ therefore cancels out at each timestep, regardless of the history. Hence they will not contribute to fluctuations in $D[t]$.

- uncorrelated, e.g., $-1-1-1-1$ and $-1-1+1+1$. For example, any two agents using the strategies $-1-1-1-1$ and $-1-1+1+1$, respectively, would take the opposite action for two of the four histories, while they would take the same action for the remaining two histories. If the $m=2$ histories occur equally often, the actions of the two agents will be uncorrelated on average.

A convenient measure of the distance (i.e., closeness) of any two strategies is the Hamming distance which is defined as the number of bits that need to be changed in going from one strategy to another. For example, the Hamming distance between $-1-1-1-1$ and $+1+1+1+1$ is 4, while the Hamming distance between $-1-1-1-1$ and $-1-1+1+1$ is just 2. Although there are $2^P \equiv 2^{2^{m=2}} \equiv 16$ strategies in the $m=2$ strategy space, it can be seen that one can choose subsets such that any strategy-pair within this subset is either anticorrelated or uncorrelated. Consider, for example, the two groups

$$U_{m=2} \equiv \{-1-1-1-1,\ +1+1-1-1,\ +1-1+1-1,\ -1+1+1-1\}, \quad (2)$$

and

$$\overline{U_{m=2}} \equiv \{+1+1+1+1,\ -1-1+1+1,\ -1+1-1+1,\ +1-1-1+1\}. \quad (3)$$

Any two strategies within $U_{m=2}$ are uncorrelated since they have a Hamming distance of 2. Likewise any two strategies within $\overline{U_{m=2}}$ are uncorrelated since they have a relative Hamming distance of 2. However, each strategy in $U_{m=2}$ has an anticorrelated strategy in $\overline{U_{m=2}}$: for example, $-1-1-1-1$ is anticorrelated to $+1+1+1+1$, $+1\ \ +1-1-1$ is anti-correlated to $-1-1+1+1$ etc. This subset of strategies comprising $U_{m=2}$ and $\overline{U_{m=2}}$, forms a Reduced Strategy Space (RSS) [14]. Since it contains the essential correlations of the Full Strategy Space (FSS), running a given game simulation within the RSS is likely to reproduce the main features obtained using the FSS [14]. The RSS has a smaller number of strategies $2.2^m = 2P \equiv 2^{m+1}$ than the FSS which has $2^P = 2^{2^m}$. For $m = 2$, there are 8 strategies in the RSS compared to 16 in the FSS, whilst for $m = 8$, there are $1.16 \times 10^{77}$ strategies in the FSS but only 512 strategies in the RSS. We note that the choice of the RSS is not unique, i.e., within a given FSS there are many possible choices for a RSS. In particular, it is possible to create $2^{2^m} / 2^{m+1}$ distinct reduced strategy spaces from the FSS. In short, the RSS provides a minimal set of strategies which 'span' the FSS and are hence representative of its full structure.

The history $\mu$ of recent outcomes changes in time, i.e., it is a dynamical variable. The history dynamics can be represented on a directed graph (a so-called digraph). The particular form of directed graph is called a de Bruijn graph. Figure 3 shows some examples of the de Bruijn graph for $m = 1, 2,$ and 3. The probability that the outcome at time $t+1$ will be a 1 (or 0) depends on the state at time *t*. Hence it will depend on the previous *m* outcomes, i.e., it depends on the particular state of the history bit-string (see also [32, 33]).

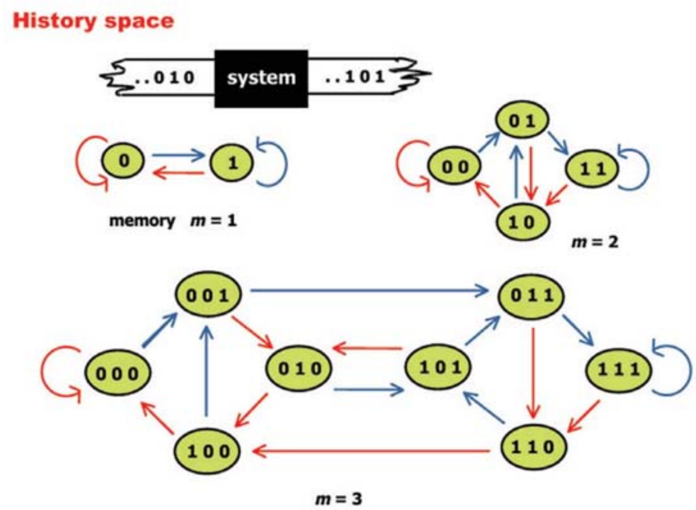


**Figure 3.** History Space. Examples of the de Bruijn graph for $m = 1$, 2, and 3. Red transitions between states correspond to the most recent global outcome 0. Blue transitions between states correspond to the most recent global outcome 1. Adapted from [43].

The initial strategy allocation among agents can be described in terms of a tensor $\Omega$ [34]. This tensor $\Omega$ describes the distribution of strategies among the $N$ individual agents. If this strategy allocation is fixed from the beginning of the game, then it acts as a quenched disorder in the system. The rank of the tensor $\Omega$ is given by the number of strategies $S$ that each agent holds. For example, for $S = 3$, the element $\Omega_{i,j,k}$ gives the number of agents assigned strategy $i$, then strategy $j$, and then strategy $k$, in that order. Hence

$$\sum_{i,j,k,\ldots}^{X} \Omega_{i,j,k,\ldots} = N, \tag{4}$$

where the value of $X$ represents the number of distinct strategies that exist within the strategy space chosen: $X = 2^{2^m}$ in the FSS, and $X = 2.2^m$ in the RSS. Figure 4 shows an example distribution $\Omega$ for $N = 101$ agents in the case of $m = 2$ and $S = 2$, in the reduced strategy space RSS. We note that a single $\Omega$ 'macrostate' corresponds to many possible 'microstates' describing the specific partitions of strategies among the agents. For example, consider an $N = 2$ agent system with $S = 1$: the microstate $(R, R')$ in which agent 1 has strategy $R$ while agent 2 has strategy $R'$, belongs to the same macrostate $\Omega$ as $(R', R)$ in which agent 1 has strategy $R'$ while agent 2 has strategy $R$. Hence the present Crowd-Anticrowd theory retained at the level of a given $\Omega$, describes the set of all games which belong to that same $\Omega$ macrostate. We also note that although $\Omega$ is not symmetric, it can be made so since the agents will typically not distinguish between the order in which the two strategies are picked. Given this, we will henceforth focus on $S = 2$ and consider the symmetrized version of the strategy allocation matrix given by $\Psi = \frac{1}{2}(\Omega + \Omega^T)$.

first strategy $R$ →

second strategy $R'$ ↓

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 4 | 2 | 1 | 2 | 1 | 2 |
| 4 | 3 | 3 | 0 | 4 | 1 | 1 | 3 |
| 2 | 3 | 1 | 1 | 2 | 1 | 3 | 0 |
| 0 | 1 | 1 | 4 | 4 | 0 | 1 | 2 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 2 | 0 | 3 | 0 | 0 | 3 |
| 1 | 0 | 1 | 1 | 2 | 1 | 2 | 0 |
| 4 | 0 | 0 | 1 | 2 | 5 | 5 | 2 |

**Figure 4.** Example distribution for the tensor $\Omega$ describing the strategy allocation for $N = 101$ agents in the case of $m = 2$ and $S = 2$, for the reduced strategy space RSS. Adapted from [43].

## 4. Crowd-Anticrowd Formalism

In addition to the excess demand $D[t]$ in such B-A-R systems, one is typically also interested in higher order-moments of this quantity – for example, the standard deviation of $D[t]$ (or 'volatility' in a financial context). This gives a measure of the fluctuations in the system, and hence can be used as a measure of 'risk' in the system. Consider an arbitrary timestep $t$ during a run of the game, at which the particular realization of the strategy allocation matrix is given by $\Psi$. There is a current score-vector $\underline{S}[t]$ and a current history $\mu[t]$ which define the state of the game. The excess demand $D[t] = D[\underline{S}[t], \mu[t]]$ is given by Equation (1). The standard deviation of $D[t]$ for this given run, corresponds to a time-average for a given realization of $\Psi$ and a given set of initial conditions. Equation (1) can be rewritten by summing over the RSS as follows:

$$D[\underline{S}[t], \mu[t]] = n_{+1}[t] - n_{-1}[t] \equiv \sum_{R=1}^{2P} a_R^{\mu[t]} n_R^{\underline{S}[t]}, \tag{5}$$

where $P = 2^m$. The quantity $a_R^{\mu[t]} = \pm 1$ is the response of strategy $R$ to the history bit-string $\mu$ at time $t$. The quantity $n_R^{\underline{S}[t]}$ is the number of agents using strategy $R$ at time $t$. The superscript $\underline{S}[t]$ is a reminder that this number of agents will depend on the strategy score at time $t$. We use the notation $\langle X[t] \rangle_t$ to denote a time-average over the variable $X[t]$ for a given $\Psi$. Hence

$$\begin{aligned} \langle D[\underline{S}[t], \mu[t]] \rangle_t &= \sum_{R=1}^{2P} \left\langle a_R^{\mu[t]} n_R^{\underline{S}[t]} \right\rangle_t \\ &= \sum_{R=1}^{2P} \left\langle a_R^{\mu[t]} \right\rangle_t \left\langle n_R^{\underline{S}[t]} \right\rangle_t, \end{aligned} \tag{6}$$

where we have used the property that $a_R^{\mu[t]}$ and $n_R^{\underline{S}[t]}$ are uncorrelated. We now consider the special case in which all histories are visited equally on average: this may arise as the result of a periodic cycling through the history space (e.g., a Eulerian trail around the de Bruijn graph) or if the histories are visited randomly. Under the property of equal histories, we can write

$$\begin{aligned}
\langle D[\underline{S}[t],\, \mu[t]]\rangle_t &= \sum_{R=1}^{2P} \left( \frac{1}{P} \sum_{\mu=0}^{P-1} a_R^{\mu[t]} \right) \left\langle n_R^{\underline{S}[t]} \right\rangle_t \\
&= \sum_{R=1}^{P} \left( \frac{1}{P} \sum_{\mu=0}^{P-1} a_R^{\mu[t]} + a_{\bar{R}}^{\mu[t]} \right) \left\langle n_R^{\underline{S}[t]} \right\rangle_t \\
&= \sum_{R=1}^{P} 0 . \left\langle n_R^{\underline{S}[t]} \right\rangle_t \\
&= 0, \qquad (7)
\end{aligned}$$

where we have used the exact result that $a_R^{\mu[t]} = -a_{\bar{R}}^{\mu[t]}$ for all $\mu[t]$, and the approximation $\left\langle n_R^{\underline{S}[t]} \right\rangle_t = \left\langle n_{\bar{R}}^{\underline{S}[t]} \right\rangle_t$. Then the average number playing each strategy is approximately equal and hence $\left\langle n_R^{\underline{S}[t]} \right\rangle_t = \left\langle n_{\bar{R}}^{\underline{S}[t]} \right\rangle_t$.

The variance of $D[t]$, which is the square of the standard deviation, is given by

$$\sigma_\Psi^2 = \left\langle D[\underline{S}[t],\, \mu[t]]^2 \right\rangle_t - \langle D[\underline{S}[t],\, \mu[t]]\rangle_t^2. \qquad (8)$$

For simplicity, we will assume the game output is unbiased and hence we can set $\langle D[\underline{S}[t],\, \mu[t]]\rangle_t = 0$. Hence

$$\sigma_\Psi^2 = \left\langle D[\underline{S}[t],\, \mu[t]]^2 \right\rangle_t$$

$$= \sum_{R,\,R'=1}^{2P} \left\langle a_R^{\mu[t]} n_R^{\underline{S}[t]} a_{R'}^{\mu[t]} n_{R'}^{\underline{S}[t]} \right\rangle_t . \tag{9}$$

In the case that the system visits all possible histories equally, the double sum can usefully be broken down into three parts, based on the correlations between the strategies: $\underline{a_R}.\underline{a_{R'}} = P$ (fully correlated), $\underline{a_R}.\underline{a_{R'}} = -P$ (fully anti-correlated), and $\underline{a_R}.\underline{a_{R'}} = 0$ (fully uncorrelated) where $\underline{a_R}$ is a vector of dimension $P$ with $R$-th component $a_R^{\mu[t]}$. This decomposition is exact in the RSS in which we are working. The equal-histories case yields

$$\begin{aligned}
\sigma_\Psi^2 &= \sum_{R=1}^{2P} \left\langle \left(a_R^{\mu[t]}\right)^2 \left(n_R^{\underline{S}[t]}\right)^2 \right\rangle_t + \sum_{R=1}^{2P} \left\langle a_R^{\mu[t]} a_{\bar{R}}^{\mu[t]} n_R^{\underline{S}[t]} n_{\bar{R}}^{\underline{S}[t]} \right\rangle_t \\
&\quad + \sum_{R \neq R' \neq \bar{R}}^{2P} \left\langle a_R^{\mu[t]} a_{R'}^{\mu[t]} n_R^{\underline{S}[t]} n_{R'}^{\underline{S}[t]} \right\rangle_t \\
&= \sum_{R=1}^{2P} \left\langle \left(n_R^{\underline{S}[t]}\right)^2 - n_R^{\underline{S}[t]} n_{\bar{R}}^{\underline{S}[t]} \right\rangle_t + \sum_{R \neq R' \neq \bar{R}}^{2P} \left\langle a_R^{\mu[t]} a_{R'}^{\mu[t]} \right\rangle_t \left\langle n_R^{\underline{S}[t]} n_{R'}^{\underline{S}[t]} \right\rangle_t \\
&= \sum_{R=1}^{2P} \left\langle \left(n_R^{\underline{S}[t]}\right)^2 - n_R^{\underline{S}[t]} n_{\bar{R}}^{\underline{S}[t]} \right\rangle_t \\
&\quad + \sum_{R \neq R' \neq \bar{R}}^{2P} \left( \frac{1}{P} \sum_{\mu=0}^{P-1} a_R^{\mu[t]} a_{\bar{R}'}^{\mu[t]} \right) \left\langle n_R^{\underline{S}[t]} n_{R'}^{\underline{S}[t]} \right\rangle_t \\
&= \sum_{R=1}^{2P} \left\langle \left(n_R^{\underline{S}[t]}\right)^2 - n_R^{\underline{S}[t]} n_{\bar{R}}^{\underline{S}[t]} \right\rangle_t .
\end{aligned} \tag{10}$$

This sum over $2P$ terms can be written equivalently as a sum over $P$ terms,

$$
\begin{aligned}
\sigma_\Psi^2 &= \sum_{R=1}^{2P} \left\langle \left(n_R^{S[t]}\right)^2 - n_R^{S[t]} n_{\overline{R}}^{S[t]} \right\rangle_t \\
&= \sum_{R=1}^{P} \left\langle \left(n_R^{S[t]}\right)^2 - n_R^{S[t]} n_{\overline{R}}^{S[t]} + \left(n_{\overline{R}}^{S[t]}\right)^2 - n_{\overline{R}}^{S[t]} n_R^{S[t]} \right\rangle_t \\
&= \sum_{R=1}^{P} \left\langle \left(n_R^{S[t]}\right)^2 - 2 n_R^{S[t]} n_{\overline{R}}^{S[t]} + \left(n_{\overline{R}}^{S[t]}\right)^2 \right\rangle_t \\
&= \sum_{R=1}^{P} \left\langle \left(n_R^{S[t]} - n_{\overline{R}}^{S[t]}\right)^2 \right\rangle_t \\
&\equiv \left\langle \sum_{R=1}^{P} \left(n_R^{S[t]} - n_{\overline{R}}^{S[t]}\right)^2 \right\rangle_t . \qquad (11)
\end{aligned}
$$

The values of $n_R^{S[t]}$ and $n_{\overline{R}}^{S[t]}$ for each $R$ will depend on the precise form of $\Psi$. The ensemble-average is denoted as $\langle \ldots \rangle_\Psi$, and for simplicity the notation $\langle \sigma_\Psi^2 \rangle_\Psi = \sigma^2$ is defined. This ensemble-average is performed on either side of Equation (11),

$$
\sigma^2 = \left\langle \left\langle \sum_{R=1}^{P} \left(n_R^{S[t]} - n_{\overline{R}}^{S[t]}\right)^2 \right\rangle_t \right\rangle_\Psi , \qquad (12)
$$

yielding the variance in the excess demand $D[t]$.

To evaluate Equation (12) analytically, we first relabel the strategies. Specifically, the sum in Equation (12) is rewritten to be over a *virtual-point ranking* $K$ and not the decimal form $R$. Consider the variation in points for a given strategy, as a function of time for a given realization of $\Psi$. The ranking (i.e., label) of a given strategy in terms of virtual-points score will typically change in time since the individual strategies have a variation in virtual-points which also varies in time. This implies that the specific identity of the '$K$-th highest-scoring strategy' changes frequently in time. It also implies that $n_{R}^{S[t]}$ varies considerably in time. Therefore in order to proceed, we shift the focus onto the time-evolution of the highest-scoring strategy, second highest-scoring strategy etc. This should have a much smoother time-evolution than the time-evolution for a given strategy. In short, the focus is shifted from the time-evolution of the virtual points of a given strategy (i.e., from $S_R[t]$) to the time-evolution of the virtual points of the $K$-th highest scoring strategy (i.e., to $S_K[t]$).

Figure 5 shows a schematic representation of how the scores of the two top scoring strategies might vary, using the new virtual-point ranking scheme. Also shown are the lowest-scoring two strategies, which at every timestep are obviously just the anticorrelated partners of the instantaneously highest-scoring two strategies. In the case that the strategies all start off with zero points, these anticorrelated strategies appear as the mirror-image, i.e., $S_K[t] = -S_{\overline{K}}[t]$. The label $K$ is used to denote the rank in terms of strategy score, i.e., $K = 1$ is the highest scoring strategy position, $K = 2$ is the second highest-scoring strategy position etc. with

$$S_{K=1} > S_{K=2} > S_{K=3} > S_{K=4} > \dots \tag{13}$$

assuming no strategy-ties. (Whenever strategy ties occur, this ranking gains a 'degeneracy' in that $S_K = S_{K+1}$ for a given $K$). A given strategy, e.g., $-1-1-1-1$, may at a given timestep have label $K = 1$, while a few timesteps later have label $K = 5$. Given that $S_R = -S_{\overline{R}}$ (i.e., all strategy scores start off at zero), then we know that $S_K = -S_{\overline{K}}$. Equation (12) can hence be rewritten exactly as

$$\sigma^2 = \left\langle \left\langle \sum_{K=1}^{P} \left( n_{K}^{S[t]} - n_{\overline{K}}^{S[t]} \right)^2 \right\rangle_t \right\rangle_\Psi . \tag{14}$$

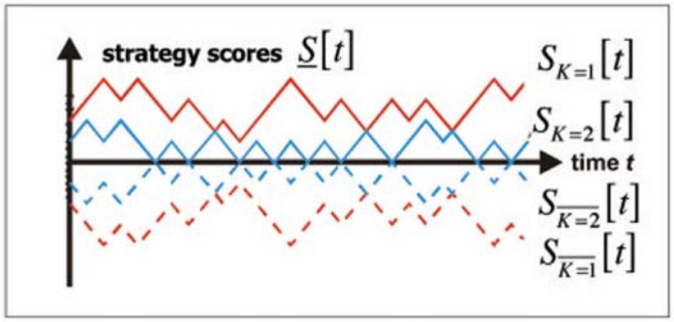


**Figure 5.** Schematic diagram of a fairly typical variation in strategy scores, as a function of time, for a competitive game. This behaviour is particularly relevant for the low $m$ regime where there are many more agents than strategies, and hence strategy rankings change in time due to being overplayed. The strategies at any given timestep can be ranked in terms of virtual-point ranking $K$, with $K = 1$ as the highest-scoring and $\overline{K = 1}$ as the lowest-scoring. The actual identity of the strategy in rank $K$ changes as time progresses, as can be seen. Ignoring accidental ties in score, there is a well defined ranking of strategies at each timestep in terms of their $K$ values. Adapted from [43].

In the systems of interest the agents are typically playing their highest-scoring strategies, hence the relevant quantity in determining how many agents will instantaneously play a given strategy, is a knowledge of its relative ranking – not the actual value of its virtual points score. This suggests that the quantities $n_K^{S[t]}$ and $n_{\overline{K}}^{S[t]}$ will fluctuate relatively little in time, and that we should now develop the problem in terms of time-averaged values.

We can determine the actual number of agents $n_K^{S[t]}$ playing the $K$-th ranked strategy at timestep $t$, from knowledge of the strategy allocation matrix $\Psi$ and the strategy scores $S[t]$, i.e., we calculate how many agents hold the $K$-th ranked strategy but do not hold another strategy with higher-ranking. The heterogeneity in the population represented by $\Psi$, combined with the strategy scores $S[t]$, determines $n_K^{S[t]}$ for each $K$

and hence the standard deviation in $D[t]$. We rewrite the number of agents playing the strategy in position $K$ at any timestep $t$, in terms of some constant value $n_K$ plus a fluctuating term :

$$n_{\overline{K}}^{S[t]} = n_K + \varepsilon_K[t]. \tag{15}$$

Consider a given timestep $t$ in the game's evolution. Suppose that an agent holds two strategies $R$ and $R'$, and at timestep $t$ they occupy ranks $K$ and $K+1$, respectively. Let the number of agents playing strategy $R$, the $K$-th ranked strategy, be $n_K$ at timestep $t$. Similarly the number of agents playing strategy $R'$, the $(K+1)$-th ranked strategy, is $n_{K+1}$ at timestep $t$. Hence $\varepsilon_K[t] = 0$ and $\varepsilon_{K+1}[t] = 0$ for this timestep, and indeed for all subsequent $t$ until the next strategy-tie. This ignore the stochatsic coin-toss for resolving strategies that are equal scoring, hence in this way we tend to overestimate the number of people playing strategies which are higher-ranking (i.e., small $K$) and underestimate the number playing strategies which are lower-ranking (i.e., large $K$). This means that we will overestimate the size of Crowds, and underestimate the size of Anticrowds. Hence we overestimate the value of the standard deviation of demand, and as a result the analytic expression $\sigma^{\text{delta } f}$ that we will derive, slightly overestimates the actual value, and hence can be regarded as an approximate *upper*-bound as shown later in Figure 7. Accounting for the correct fraction of degenerate vs. non-degenerate timesteps, will yield a more accurate calculation of the time-average $\langle \ldots \rangle_t$ in Equation (14).

For the moment, we assume that we can choose a suitable constant $n_K$ such that the fluctuation $\varepsilon_K[t]$ represents a small noise term. Hence,

$$\begin{aligned}
\sigma^2 &= \left\langle \sum_{K=1}^{P} \left\langle \left[ n_K + \varepsilon_K[t] - n_{\overline{K}} - \varepsilon_{\overline{K}}[t] \right]^2 \right\rangle_t \right\rangle_\Psi \\
&= \left\langle \sum_{K=1}^{P} \left\langle \left[ (n_K - n_{\overline{K}}) + (\varepsilon_K[t] - \varepsilon_{\overline{K}}[t]) \right]^2 \right\rangle_t \right\rangle_\Psi
\end{aligned}$$

$$\approx \left\langle \sum_{K=1}^{P} \left\langle [n_K - n_{\overline{K}}]^2 \right\rangle_t \right\rangle_\Psi = \left\langle \sum_{K=1}^{P} [n_K - n_{\overline{K}}]^2 \right\rangle_\Psi , \tag{16}$$

since the latter two terms involving noise will average out to be small. The resulting expression in Equation (16) involves no time dependence. The averaging over $\Psi$ can then be taken inside the sum. The individual terms in the sum, i.e., $\left\langle [n_K - n_{\overline{K}}]^2 \right\rangle_\Psi$, are just an expectation value of a function of two variables $n_K$ and $n_{\overline{K}}$. Each term can therefore be rewritten exactly using the joint probability distribution for $n_K$ and $n_{\overline{K}}$, which we shall call $P(n_K, n_{\overline{K}})$. Hence

$$\sigma^2 = \sum_{K=1}^{P} \left\langle [n_K - n_{\overline{K}}]^2 \right\rangle_\Psi$$

$$= \sum_{K=1}^{P} \sum_{n_K=0}^{N} \sum_{n_{\overline{K}}=0}^{N} [n_K - n_{\overline{K}}]^2 P(n_K, n_{\overline{K}}), \tag{17}$$

where the standard probability result involving functions of two variables has been used.

The question then arises of how to evaluate Equation (17). In general, its value will depend on the detailed form of the joint probability function $P(n_K, n_{\overline{K}})$ which in turn will depend on the ensemble of quenched disorders $\{\Psi\}$ which are being averaged over. We start off by looking at Equation (17) in the limiting case where the averaging over the quenched disorder matrix is dominated by matrices $\Psi$ which are nearly at. This will be a good approximation in the 'crowded' limit of small $m$ in which there are many more agents than available strategies, since the standard deviation of an element in $\Psi$ (i.e., the standard deviation in bin-size) is then much smaller than the mean bin-size. [N.B. If $\Omega$ is approximately at, then so is $\Psi$]. In this limiting case, there are several nice features:

• Given the ranking in terms of virtual-points, i.e., $S_{K=1} > S_{K=2} > S_{K=3} > S_{K=4} > \ldots$ which holds by definition of the labels $\{K\}$ if we neglect tie-breaks, we will also have

$$n_{K=1} > n_{K=2} > n_{K=3} > n_{K=4} > \ldots. \tag{18}$$

Hence the rankings in terms of highest virtual-points and popularity are identical. By contrast, the ordering in terms of the labels $\{R\}$ would not be sequential, i.e., it is *not* true that $n_{R=1} > n_{R=2} > n_{R=3} > n_{R=4} > \ldots.$

• The strategy $\overline{K}$, which is anticorrelated to strategy $K$, occupies position $\overline{K} = 2P + 1 - K$ in this popularity-ranked list.

• The probability distribution $P(n_K, n_{\overline{K}})$ will be sharply peaked around the $n_K$ and $n_{\overline{K}}$ values given by the mean values for a flat quenched-disorder matrix $\Psi$. We will label these mean values as $\overline{n_K}$ and $\overline{n_{\overline{K}}}$.

The last point implies that $P(n_K, n_{\overline{K}}) = \delta_{n_K, \overline{n_K}} \delta_{n_{\overline{K}}, \overline{n_{\overline{K}}}}$ and so

$$\sigma^2 = \sum_{K=1}^{P} [\overline{n_K} - \overline{n_{\overline{K}}}]^2. \tag{19}$$

We note that there is a very simple interpretation of Equation (19). It represents the sum of the variances for each Crowd-Anticrowd pair. For a given strategy $K$ there is an anticorrelated strategy $\overline{K}$. The $\overline{n_K}$ agents using strategy $K$ are doing the *opposite* of the $\overline{n_{\overline{K}}}$ agents using strategy $\overline{K}$ *irrespective* of the history bit-string. Hence the effective group-size for each Crowd-Anticrowd pair is $n_K^{eff} = \overline{n_K} - \overline{n_{\overline{K}}}$ : this represents the net step-size $d$ of the Crowd-Anticrowd pair in a random-walk contribution to the total variance. Hence, the net contribution by this Crowd-Anticrowd pair to the variance is given by

$$[\sigma^2]_{K\overline{K}} = 4pqd^2 = 4.\frac{1}{2}.\frac{1}{2}[n_K^{eff}]^2 = [\overline{n_K} - \overline{n_{\overline{K}}}]^2, \tag{20}$$

where $p = q = 1/2$ for a random walk. Since all the strong correlations have been removed (i.e., anti-correlations) it can therefore be assumed that the separate Crowd-Anticrowd pairs execute random walks which are *uncorrelated* with respect to each other. Hence the total variance is given by the sum of the individual variances,

$$\sigma^2 = \sum_{K=1}^{P} [\sigma^2]_{K\overline{K}} = \sum_{K=1}^{P} [\overline{n_K} - \overline{n_{\overline{K}}}]^2, \tag{21}$$

which corresponds exactly to Equation (19). If strategy-ties occur frequently, then one has to be more careful about evaluating $\overline{n_K}$ since its value may be affected by the tie-breaking rule.

Each element of $\Psi$ has a mean of $N/(2P)^S$ agents per 'bin'. In the case of small $m$ and hence densely-filled $\Psi$, the fluctuations in the number of agents per bin will be small compared to this mean value. For the case $S = 2$, the mean number of agents whose highest scoring strategy is the strategy occupying position $K$ at timestep $t$, will therefore be given by summing the appropriate rows and columns of this quenched disorder matrix $\Psi$. Figure 6 provides a schematic representation of $\Psi$ with $m = 2$, $s = 2$, in the RSS. If the matrix $\Psi$ is flat, then any re-ordering due to changes in the strategy ranking has no effect on the form of the matrix. Therefore, the number of agents playing the $K$-th highest-scoring strategy, will always be proportional to the number of shaded bins at that $K$ (see Figure 6 for $K = 3$). The shaded elements in Figure 6 therefore represent those agents who hold a strategy that is ranked third highest in score, i.e., $K = 3$. In games where the agents use their highest scoring strategy, any agent using the strategy in position $K = 3$ cannot have any strategy with a higher position. Hence the agents using the strategy in position $K = 3$ must lie in one of the shaded bins. Since it is assumed that the coverage of the bins is uniform, the mean number of agents using the strategy in position $K = 3$ is given by

$$\overline{n_{K=3}} = N.\frac{1}{(2P)^2}\sum(shaded\ bins)$$

$$= N.\frac{1}{64}.[(8-3)+(8-3)+1]$$

$$= \frac{11}{64}N. \quad (22)$$

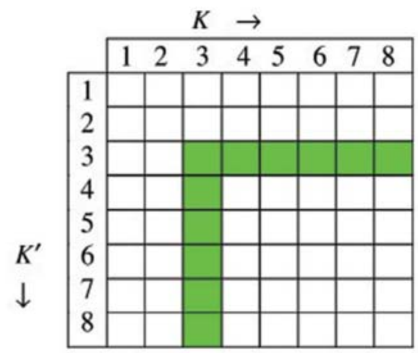


**Figure 6.** Schematic representation of the strategy allocation matrix $\Psi$ with $m = 2$ and $s = 2$, in the RSS. The strategies are ranked according to strategy score, and are labelled by the rank *K*. In the limit that $\Psi$ is essentially flat, then the number of agents playing the *K*-th highest-scoring strategy, is just proportional to the number of shaded bins at that *K*. Adapted from [43].

For more general *m* and *s* values this becomes

$$\overline{n_K} = \frac{N}{(2P)^2}\Big[S(2P-K)^{S-1} + \frac{S(S-1)}{2}(2P-K)^{S-2} + \ldots + 1\Big]$$

$$= \frac{N}{(2P)^S} \sum_{r=0}^{S-1} \frac{S!}{(S-r)!\,r!} [2P - K]^r$$

$$= \frac{N}{(2P)^S} \left([2P - K + 1]^S - [2P - K]^S\right)$$

$$= N.\left(\left[1 - \frac{(K-1)}{2P}\right]^S - \left[1 - \frac{K}{2P}\right]^S\right), \tag{23}$$

with $P \equiv 2^m$. In the case where each agent holds two strategies, $S = 2$, $\overline{n_K}$ can be simplified to

$$\overline{n_K} = N.\left(\left[1 - \frac{(K-1)}{2P}\right]^2 - \left[1 - \frac{K}{2P}\right]^2\right)$$

$$= \frac{(2^{m+2} - 2K + 1)}{2^{2(m+1)}} N. \tag{24}$$

Similarly for $\overline{n_{\overline{K}}}$, the simplification is as follows:

$$\overline{n_{\overline{K}}} = \frac{(2^{m+2} - 2\overline{K} + 1)}{2^{2(m+1)}} N = \frac{(2K-1)}{2^{2(m+1)}} N, \tag{25}$$

where the relation $\overline{K} = 2P - K + 1 \equiv 2^{m+1} - K + 1$ has been used. It is emphasized that these results depend on the assumption that the averages are dominated by the effects of at distributions of the quenched disorder matrix $\Psi$, and hence will only be quantitatively valid for low $m$. Using Equations (24) and (25) in Equation (19) gives

$$\sigma^2 = \sum_{K=1}^{P} \left[\frac{(2^{m+2} - 2K + 1)}{2^{2(m+1)}} N - \frac{(2K-1)}{2^{2(m+1)}} N\right]^2$$

$$= \frac{N^2}{2^{2(2m+1)}} \sum_{K=1}^{P} [2^{m+1} - 2K + 1]^2$$

$$= \frac{N^2}{3.2^m}\left(1 - 2^{-2(m+1)}\right), \tag{26}$$

and hence

$$\sigma^{\text{delta } f} = \frac{N}{\sqrt{3}.2^{m/2}}\left(1 - 2^{-2(m+1)}\right)^{\frac{1}{2}}, \tag{27}$$

which is valid for small $m$. (The rationale behind the choice of superscript 'delta $f$' will become apparent shortly.) This derivation has assumed that there are no strategy ties – more precisely, we have assumed that the game rules governing strategy ties do not upset the identical forms of the rankings in terms of highest virtual points and popularity. As discussed earlier, this tends to overestimate the size of the Crowds using high-ranking strategies, and underestimate the size of the Anticrowds using low-ranking strategies. Hence the Crowd-Anticrowd cancellation is underestimated, and consequently $\sigma^{\text{delta } f}$ will overestimate the actual $\sigma$ value. As we will see later in Figure 7, $\sigma^{\text{delta } f}$ does indeed act as an approximate upper bound.

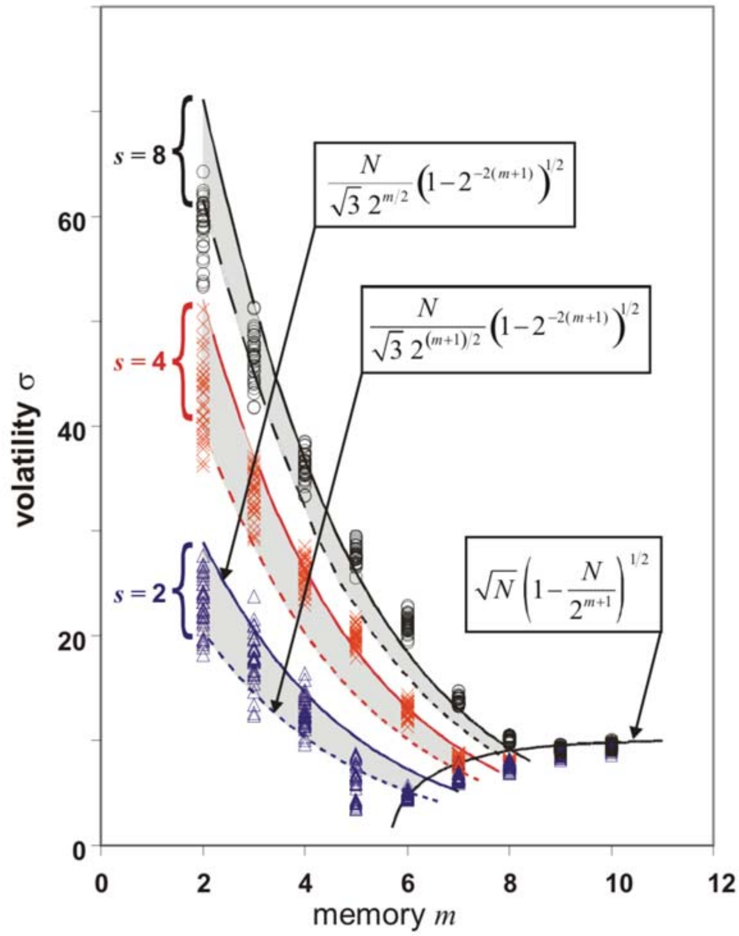


**Figure 7.** Crowd-Anticrowd theory vs. numerical simulation results for Minority Game as a function of memory size $m$, for $N = 101$ agents, at $S = 2,\ 4,$ and 8. At each $S$ value, analytic forms of standard deviation in excess demand $D[t]$, are shown corresponding to $\sigma^{\text{delta } f}$ (upper solid line), $\sigma^{\text{flat } f}$ (lower dashed line) and $\sigma^{\text{flat } f,\text{ high } m}$ (monotonically-increasing solid line which is independent of $S$). The numerical values were obtained from different simulation runs (triangles, crosses and circles). The corresponding result for a null, coin-toss (i.e., random) model is $\sqrt{N}$ for all $m$ (i.e., $\sqrt{101}$ in this figure, which is approximately 10). Hence the emergent property of large changes at small $m$ is much larger, scaling instead as $N$ not $\sqrt{N}$. Adapted from [43].

A similar calculation can be performed for a non-flat quenched disorder matrix $\Psi$, at small $m$, and also the non-flat quenched disorder matrix $\Psi$ at large $m$. For full details, we refer to [43]. The standard deviation $\sigma$ of the excess demand $D[t]$ is shown in Figure 7, for the basic Minority Game. The spread in numerical values from individual runs, for a given $m$, indicates the extent to which the choice of $\Psi$ alters the dynamics of the MG. The upper line for each $S$ value at low $m$, is Equation (27) showing $\sigma^{\text{delta } f}$. The analytic expressions capture the strong correlations and hence the fluctuations in the system [43].

## 5. Application: Alloy Minority Game

Consider a population containing some agents with memory $m_1$, and some agents with memory $m_2$, where $m_1 < m_2$. For a pure population of agents with the same memory $m_1$, there is information left in the history time-series [23]. In the small $m_1$ limit, however, this information is hidden in bit-strings of length greater than $m_1$ and hence is not accessible to these agents. However, it would in principle be accessible to agents with a larger memory $m_2$. In the mixed population or 'alloy', there are two sub-populations comprising $N_{m_1}$ agents with memory $m_1$ and $S_1$ strategies per agent, and $N_{m_2} = N - N_{m_1}$ agents with memory $m_2$ and $S_2$ strategies per agent. Let us focus on the variance $\sigma^2$ of demand $D[t]$. If each population were pure, it would form Crowd-Anticrowd groups as discussed earlier in this paper, and hence provide a variance given by the sum of the variances of these uncorrelated groups. In the mixed population, we can assume as a first approximation that the individual groups for different $m$ are also uncorrelated. In short, the agents looking at patterns of length $m_1$, and the agents looking at patterns of length $m_2$, act in an uncorrelated way with respect to each other. Hence the variance in the total $D[t]$ from both sub-populations, can be obtained by adding separately the contributions to the variance from the $m_1$ agents and the $m_2$ agents. Hence $\sigma^2 = \sigma_1^2 + \sigma_2^2$, where $\sigma_1(\sigma_2)$ is the variance due to the $m_1(m_2)$ agents. Defining the concentration of $m_1$ agents as $x = N_{m_1} / N$, gives $\sigma_1^2 = C^2(m_1, S_1)x^2N^2$ and $\sigma_2^2 = C^2(m_2, S_2)(1-x)^2N^2$, where the expressions for $C^2(m_1, S_1)$ and $C^2(m_2, S_2)$ follow from Equations (19) and (23):

$$C^2(m_i, S_i) = \sum_{K=1}^{2^{m_i}}\Bigg(\left[1 - \frac{(K-1)}{2^{m_i+1}}\right]^{S_i} - \left[1 - \frac{K}{2^{m_i+1}}\right]^{S_i} - \left[1 - \frac{(2^{m_i+1} - K)}{2^{m_i+1}}\right]^{S_i} + \left[1 - \frac{2^{m_i+1} + 1 - K}{2^{m_i+1}}\right]^{S_i}\Bigg)^2. \tag{28}$$

Hence

$$\sigma = N[C^2(m_1, S_1)x^2 + C^2(m_2, S_2)(1-x)^2]^{1/2}. \tag{29}$$

It can be seen that Equation (29) will generally exhibit a minimum in $\sigma$ at finite $x$, hence the mixed population uses the limited global resource more efficiently than a pure population of either $(m_1, S_1)$ or $(m_2, S_2)$ agents. This analytic result has been confirmed in numerical simulations of a mixed population [28, 35].

## 6. Conclusion

The mathematical Crowd-Anticrowd theory that we present, helps understand the emergence of large changes (i.e., large fluctuations) in competitive multi-agent systems, in particular those based on an underlying binary structure, and should have applicability for more general multi-agent systems. It also raises the intriguing possibility that conventional many-body physics might be open to re-interpretation in terms of an appropriate multi-particle ‘game’. We leave this for future work.

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