Transnational Journal of Mathematical Analysis and Applications
Vol. 5, Issue 1, 2017, Pages 1-15
ISSN 2347-9086
Published Online on March 28, 2017

http://jyotiacademicpress.org

HAMILTON-JACOBI SYSTEM OF PDES GOVERNED BY HIGHER-ORDER LAGRANGIANS

SAVIN TREANŢĂ

Department of Applied Mathematics
Faculty of Applied Sciences
University Politehnica of Bucharest
313 Splaiul Independentei
060042 – Bucharest
Romania
e-mail: savin_treanta@yahoo.com

Abstract

This paper aims to present some aspects of Hamilton-Jacobi theory involving higher-order Lagrangians. More precisely, using a non-standard Legendrian duality, we investigate: Hamilton-Jacobi PDE and Hamilton-Jacobi system of PDEs.

1. Introduction

Over time, many researchers have been interested in the study of Hamilton-Jacobi equations. It is well-known that the classical (single-time) Hamilton-Jacobi theory appeared in mechanics or in information theory from the desire to describe simultaneously the motion of a particle by a wave and the information dynamics by a wave carrying information.

2010 Mathematics Subject Classification: 70H03, 70H05, 70H20, 70H50, 49K15, 49K20.
Keywords and phrases: Hamilton-Jacobi PDE, Hamilton-Jacobi system of PDEs, generating function, non-standard Legendrian duality, multi-time.
Communicated by Lubin Georgiev Vulkov.
Received February 7, 2017; Revised March 7, 2017

Thus, the Euler-Lagrange ODEs or the associated Hamilton ODEs are replaced by PDEs that characterize the generating function. Later, using the geometric setting of k-osculator bundle, Miron [5] and Roman [8] studied the geometry of higher-order Lagrange spaces, providing some applications in mechanics and physics. Also, Krupkova [3] investigated the Hamiltonian field theory in terms of differential geometry and local coordinate formulas.

The *multi-time* version of Hamilton-Jacobi theory has been extensively studied by many researchers in the last few years (see, for instance, Rochet [7], Motta and Rampazzo [6], Cardin and Viterbo [1], Udrişte et al. [16], Treanţă [10]). The present work can be seen as a natural continuation of a recent paper (Treanţă [10]), where only multi-time Hamilton-Jacobi theory via second-order Lagrangians is considered. In this paper, we develop our points of view, by developing new concepts and methods for a theory that involves single-time and multi-time higher-order Lagrangians. For other different but connected ideas to this subject, the reader is directed to Ibragimov [2], Lebedev and Cloud [4], Treanţă and Vârsan [11], Treanţă [12], Udrişte and Ţevy [15]. This work can be used as source for research problems and it should be of interest to engineers and applied mathematicians.

2. Hamilton ODEs and Hamilton-Jacobi PDE

This section introduces Hamilton ODEs and Hamilton-Jacobi PDE based on single-time higher-order Lagrangians.

Consider $k \geq 2$ a fixed natural number, $t \in [t_0, t_1] \subseteq R$, $x : [t_0, t_1] \subseteq R \to R^n$, $x = (x^i(t))$, $i = \overline{1, n}$, and $x^{(a)}(t) := \dfrac{d^a}{dt^a} x(t)$, $a \in \{1, 2, \dots, k\}$. We shall use alternatively the index a to mark the derivation or to mark the summation. The real C^{k+1}-class function $L\left(t, x(t), x^{(1)}(t), \dots, x^{(k)}(t)\right)$, called *single-time higher-order Lagrangian*, depends by $(k+1)n+1$ variables. Denoting

$$\frac{\partial L}{\partial x^{(a)i}}\left(t,\ x(t),\ x^{(1)}(t),\ \ldots,\ x^{(k)}(t)\right) = p_{ai}(t), \quad a \in \{1, 2, \ldots, k\},$$

the link $L = x^{(a)i} p_{ai} - H$ (with summation over the repeated indices!) changes the following simple integral functional

$$I(x(\cdot)) = \int_{t_0}^{t_1} L\left(t,\ x(t),\ x^{(1)}(t),\ \ldots,\ x^{(k)}(t)\right) dt \tag{P}$$

into

$$J(x(\cdot),\ p_1(\cdot),\ \ldots,\ p_k(\cdot)) = \int_{t_0}^{t_1} \left(x^{(a)i}(t) p_{ai}(t) - H(t,\ x(t),\ p_1(t),\ \ldots,\ p_k(t))\right) dt \tag{P'}$$

and the (higher-order) Euler-Lagrange ODEs,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x^{(1)i}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial x^{(2)i}} - \ldots + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial x^{(k)i}} = 0,\ i \in \{1, 2, \ldots, n\},$$

(no summation after k) written for (P′), are just the *higher-order ODEs of Hamiltonian type*,

$$\sum_{a=1}^{k} (-1)^{a+1} \frac{d^a}{dt^a} p_{ai} = -\frac{\partial H}{\partial x^i}, \quad \frac{d^a}{dt^a} x^i = \frac{\partial H}{\partial p_{ai}}, \quad a \in \{1, 2, \ldots, k\}.$$

Applications. (a) The optimal growth problem ([13])**.** In order to formulate our study problem, let us introduce the following tools: the consumption level function $C = Y(K) - \dot{K}$, where Y is the Gross national income (thus, C is the Gross national product left over after the capital accumulation $\dot{K}$ is accomplished); the growth rate $\dot{C}$ and the utility $U(C,\ \dot{C})$. To transform the previous utility into a linear in acceleration second-order Lagrangian, it is suitable to consider $Y(K) = bK$, b=*const.*, and $U(C, \dot{C}) = C^a + \alpha\dot{C}$, where $a, \alpha \in [0, 1]$. Therefore, our study refers to maximizing the functional

$$I(K(\cdot)) = \int_0^T U\big(K(t),\ \dot{K}(t),\ \ddot{K}(t)\big)dt.$$

The necessary optimality conditions

$$ab(bK - \dot{K})^{a-1} - a(1-a)(bK - \dot{K})^{a-2}(b\dot{K} - \ddot{K}) = 0,$$

gives the solution

$$K(t) = A_1 \exp(bt) + A_2 \exp\left(\frac{bt}{1-a}\right),$$

where A_1, A_2 are constants generated by the boundary conditions $K(0) = K_0$, $K(T) = K_T$.

(b) The motion of a spinning particle ([14])**.** The motion of a particle rotating around its translating center is described by the following fourth-order differential system

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} = 0,$$

coming (differentiating two times) from

$$\frac{d^2x}{dt^2} + x = at + b,$$

with *a, b* constant vectors and $x = (x^1, x^2, x^3) \in (R^3, \delta_{ij})$. The previous fourth-order differential system arises from the second-order Lagrangian

$$L = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - \frac{1}{2}\delta_{ij}\ddot{x}^i\ddot{x}^j,$$

and it admits the first integral

$$H = \frac{1}{2}\delta^{ij}p_ip_j - \frac{1}{2}\delta^{ij}q_iq_j + \delta^{ij}p_i\dot{q}_j, \quad i, j \in \{1, 2, 3\}.$$

2.1. Hamilton-Jacobi PDE based on higher-order Lagrangians

Further, we shall describe Hamilton-Jacobi PDE governed by higher-order Lagrangians with single-time evolution variable.

Let us consider the real function $S : R \times R^{kn} \to R$ and the constant level sets $\sum_c : S\left(t, x, x^{(1)}, \dots, x^{(k-1)}\right) = c$, $k \geq 2$ a fixed natural number, where $x^{(a)}(t) := \frac{d^a}{dt^a} x(t)$, $a = \overline{1, k-1}$. We assume that these sets are hypersurfaces in R^{kn+1}, that is the normal vector field satisfies

$$\left(\frac{\partial S}{\partial t}, \frac{\partial S}{\partial x^i}, \frac{\partial S}{\partial x^{(1)i}}, \dots, \frac{\partial S}{\partial x^{(k-1)i}}\right) \neq (0, \dots, 0).$$

Let $\widetilde{\Gamma} : \left(t, x^i(t), x^{(1)i}(t), \dots, x^{(k-1)i}(t)\right)$, $t \in R$, be a transversal curve to the hypersurfaces $\sum_c$. Then, the function $c(t) = S\left(t, x(t), x^{(1)}(t), \dots, x^{(k-1)}(t)\right)$ has nonzero derivative

$$\begin{aligned}\frac{dc}{dt}(t) := L\left(t, x(t), x^{(1)}(t), \dots, x^{(k)}(t)\right) &= \frac{\partial S}{\partial t}\left(t, x(t), x^{(1)}(t), \dots, x^{(k-1)}(t)\right) \\ &+ \frac{\partial S}{\partial x^i}\left(t, x(t), x^{(1)}(t), \dots, x^{(k-1)}(t)\right) x^{(1)i}(t) \\ &+ \sum_{r=1}^{k-1} \frac{\partial S}{\partial x^{(r)i}}\left(t, x(t), x^{(1)}(t), \dots, x^{(k-1)}(t)\right) x^{(r+1)i}(t). \qquad (2.1)\end{aligned}$$

By computation, we obtain the *canonical momenta*

$$\frac{\partial L}{\partial x^{(a)i}}\left(t, x(t), x^{(1)}(t), \dots, x^{(k)}(t)\right) = \frac{\partial S}{\partial x^{(a-1)i}}\left(t, x(t), x^{(1)}(t), \dots, x^{(k-1)}(t)\right) := p_{ai}(t),$$

where $a \in \{1, 2, \dots, k\}$. In these conditions, the relations

$$x^{(a)} = x^{(a)}(t, x, p_1, \dots, p_k), \quad a \in \{1, 2, \dots, k\},$$

become

$$x^{(a)} = x^{(a)}\left(t,\ x,\ \frac{\partial S}{\partial x},\ \dots,\ \frac{\partial S}{\partial x^{(k-1)}}\right), \quad a \in \{1,\ 2,\ \dots,\ k\}.$$

On the other hand, the relation (2.1) can be rewritten as

$$-\frac{\partial S}{\partial t}\left(t,\ x(t),\ x^{(1)}(t),\ \dots,\ x^{(k-1)}(t)\right)$$

$$= \frac{\partial S}{\partial x^{i}}\left(t,\ x(t),\ x^{(1)}(t),\ \dots,\ x^{(k-1)}(t)\right)x^{(1)i}\left(t,\ x^{i},\ \frac{\partial S}{\partial x^{i}}(\cdot),\ \dots,\ \frac{\partial S}{\partial x^{(k-1)i}}(\cdot)\right)$$

$$+ \sum_{r=1}^{k-1}\frac{\partial S}{\partial x^{(r)i}}\left(t,\ x(t),\ x^{(1)}(t),\ \dots,\ x^{(k-1)}(t)\right)x^{(r+1)i}\left(t,\ x^{i},\ \frac{\partial S}{\partial x^{i}}(\cdot),\ \dots,\ \frac{\partial S}{\partial x^{(k-1)i}}(\cdot)\right)$$

$$- L\left(t,\ x(t),\ x^{(1)}(t),\ \dots,\ x^{(k)}(t)\right). \tag{2.2}$$

Definition 2.1. The Lagrangian $L\left(t,\ x(t),\ x^{(1)}(t),\ \dots,\ x^{(k)}(t)\right)$ is called *super-regular* if the system

$$\frac{\partial L}{\partial x^{(a)i}}\left(t,\ x(t),\ x^{(1)}(t),\ \dots,\ x^{(k)}(t)\right) = p_{ai}(t), \quad a \in \{1,\ 2,\ \dots,\ k\},$$

defines the function of components

$$x^{(a)} = x^{(a)}(t,\ x,\ p_1,\ \dots,\ p_k), \quad a \in \{1,\ 2,\ \dots,\ k\}.$$

The super-regular Lagrangian L enters in duality with the function of Hamiltonian type

$$H(t,\ x,\ p_1,\ \dots,\ p_k) = x^{(a)i}(t,\ x,\ p_1,\ \dots,\ p_k)\frac{\partial L}{\partial x^{(a)i}}\left(t,\ x,\ \dots,\ x^{(k)i}(t,\ x,\ p_1,\ \dots,\ p_k)\right)$$

$$- L\left(t,\ x,\ x^{(1)i}(t,\ x,\ p_1,\ \dots,\ p_k),\ \dots,\ x^{(k)i}(t,\ x,\ p_1,\ \dots,\ p_k)\right),$$

(*single-time higher-order non-standard Legendrian duality*) or, shortly,

$$H = x^{(a)i}p_{ai} - L.$$

At this moment, we can rewrite (2.2) as *Hamilton-Jacobi PDE based on higher-order Lagrangians*,

$$\frac{\partial S}{\partial t} + H\left(t,\ x^i,\ \frac{\partial S}{\partial x^i},\ \frac{\partial S}{\partial x^{(1)i}},\ \ldots,\ \frac{\partial S}{\partial x^{(k-1)i}}\right) = 0, \quad i = \overline{1,\ n}.$$

As a rule, this Hamilton-Jacobi PDE based on higher-order Lagrangians is endowed with the initial condition

$$S\left(0,\ x,\ x^{(1)},\ \ldots,\ x^{(k-1)}\right) = S_0\left(x,\ x^{(1)},\ \ldots,\ x^{(k-1)}\right).$$

The solution $S\left(t, x, x^{(1)}, \ldots, x^{(k-1)}\right)$ is called the *generating function* of the canonical momenta.

Remark 2.1. Conversely, let $S\left(t, x, x^{(1)}, \ldots, x^{(k-1)}\right)$ be a solution of the Hamilton-Jacobi PDE based on higher-order Lagrangians. We define

$$p_{ai}(t) = \frac{\partial S}{\partial x^{(a-1)i}}\left(t,\ x(t),\ x^{(1)}(t),\ \ldots,\ x^{(k-1)}(t)\right), \quad a \in \{1,\ 2,\ \ldots,\ k\}.$$

Then, the following link appears (see summation over the repeated indices!)

$$\int_{t_0}^{t_1} L\left(t,\ x(t),\ x^{(1)}(t),\ \ldots,\ x^{(k)}(t)\right)dt$$

$$= \int_{t_0}^{t_1}\left[x^{(a)i}(t)p_{ai}(t) - H\left(t,\ x^{i(t)},\ \frac{\partial S}{\partial x^i}(\cdot),\ \frac{\partial S}{\partial x^{(1)i}}(\cdot),\ \ldots,\ \frac{\partial S}{\partial x^{(k-1)i}}(\cdot)\right)\right]dt$$

$$= \int_\Gamma \frac{\partial S}{\partial x^{(a-1)i}}\,dx^{(a-1)i} + \frac{\partial S}{\partial t}\,dt = \int_\Gamma dS.$$

The last formula shows that the action integral can be written as a path independent curvilinear integral.

Theorem 2.1. *The generating function of the canonical momenta is solution of the Cauchy problem*

$$\frac{\partial S}{\partial t} + H\left(t,\ x^i,\ \frac{\partial S}{\partial x^i},\ \frac{\partial S}{\partial x^{(1)i}},\ \ldots,\ \frac{\partial S}{\partial x^{(k-1)i}}\right) = 0,$$

$$S\left(0,\ x,\ x^{(1)},\ \ldots,\ x^{(k-1)}\right) = S_0\left(x,\ x^{(1)},\ \ldots,\ x^{(k-1)}\right).$$

Theorem 2.2. *If*

$$L\left(t, x(t), x^{(1)}(t), \ldots, x^{(k)}(t)\right)$$

$$= \frac{\partial S}{\partial t}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)$$

$$+ \frac{\partial S}{\partial x^i}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)x^{(1)i}(t)$$

$$+ \sum_{r=1}^{k-1}\frac{\partial S}{\partial x^{(r)i}}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)x^{(r+1)i}(t)$$

is fulfilled and its domain is convex, then

$$\frac{\partial S}{\partial t} + H\left(t, x^i, \frac{\partial S}{\partial x^i}, \frac{\partial S}{\partial x^{(1)i}}, \ldots, \frac{\partial S}{\partial x^{(k-1)i}}\right)$$

is invariant with respect to the variable x.

Proof. By direct computation, we get

$$\frac{\partial L}{\partial x^j}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k)}(t)\right)$$

$$= \frac{\partial^2 S}{\partial t \partial x^j}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)$$

$$+ \frac{\partial^2 S}{\partial x^i \partial x^j}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)x^{(1)i}(t)$$

$$+ \sum_{r=1}^{k-1}\frac{\partial^2 S}{\partial x^{(r)i}\partial x^j}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)x^{(r+1)i}(t),$$

equivalent with

$$-\frac{\partial H}{\partial x^j}\left(t, x(t), \frac{\partial S}{\partial x}(\cdot), \frac{\partial S}{\partial x^{(1)}}(\cdot), \ldots, \frac{\partial S}{\partial x^{(k-1)}}(\cdot)\right)$$

$$= \frac{\partial^2 S}{\partial t \partial x^j}\left(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)\right)$$

$$+ \frac{\partial^2 S}{\partial x^i \partial x^j}\Big(t,\ x(t),\ x^{(1)}(t),\ \ldots,\ x^{(k-1)}(t)\Big)x^{(1)i}(t)$$

$$+ \sum_{r=1}^{k-1} \frac{\partial^2 S}{\partial x^{(r)i} \partial x^j}\Big(t,\ x(t),\ x^{(1)}(t),\ \ldots,\ x^{(k-1)}(t)\Big)x^{(r+1)i}(t),$$

or,

$$\frac{\partial}{\partial x^j}\left[\frac{\partial S}{\partial t} + H\left(t,\ x^i,\ \frac{\partial S}{\partial x^i},\ \frac{\partial S}{\partial x^{(1)i}},\ \ldots,\ \frac{\partial S}{\partial x^{(k-1)i}}\right)\right] = 0,$$

$$\frac{\partial S}{\partial t} + H\left(t,\ x^i,\ \frac{\partial S}{\partial x^i},\ \frac{\partial S}{\partial x^{(1)i}},\ \ldots,\ \frac{\partial S}{\partial x^{(k-1)i}}\right) = f(t,\ x^{(1)}(t),\ \ldots,\ x^{(k)}(t))$$

and the proof is complete.

3. Hamilton-Jacobi System of PDEs via Multi-Time Higher-Order Lagrangians

In this section, we shall introduce Hamilton-Jacobi system of PDEs governed by higher-order Lagrangians with multi-time evolution variable.

Let $S : R^m \times R^n \times R^{nm} \times R^{nm(m+1)/2} \times \cdots \times R^{[nm(m+1)\cdot \ldots \cdot (m+k-2)]/(k-1)!} \to R$ be a real function and the constant level sets,

$$\sum\nolimits_c : S(t,\ x,\ x_{\alpha_1},\ \ldots,\ x_{\alpha_1 \ldots \alpha_{k-1}}) = c,$$

where $k \geq 2$ is a fixed natural number, $t = (t^1,\ \ldots,\ t^m) \in R^m$, $x_{\alpha_1} := \frac{\partial x}{\partial t^{\alpha_1}},\ \ldots,\ x_{\alpha_1 \ldots \alpha_{k-1}} := \frac{\partial^{k-1} x}{\partial t^{\alpha_1} \ldots \partial t^{\alpha_{k-1}}}$. Here $\alpha_j \in \{1,\ 2, \ldots,\ m\}$, $j = \overline{1,\ k-1}$, $x = (x^1,\ \ldots,\ x^n) = (x^i)$, $i \in \{1,\ 2,\ \ldots,\ n\}$. We assume that these sets are submanifolds in $R^{m+n+\ldots+[nm(m+1)\cdot \ldots \cdot (m+k-2)]/(k-1)!}$. Consequently, the normal vector field must satisfy

$$\left(\frac{\partial S}{\partial t^\beta},\ \frac{\partial S}{\partial x^i},\ \frac{\partial S}{\partial x^i_{\alpha_1}},\ \ldots,\ \frac{\partial S}{\partial x^i_{\alpha_1 \ldots \alpha_{k-1}}}\right) \neq (0,\ \ldots,\ 0).$$

Let $\widetilde{\Gamma} : \left(t,\ x^i(t),\ x^i_{\alpha_1}(t),\ \ldots,\ x^i_{\alpha_1 \ldots \alpha_{k-1}}(t)\right),\ t \in R^m,$ be an m-sheet transversal to the submanifolds $\sum_c$. Then, the real function

$$c(t) = S(t,\ x(t),\ x_{\alpha_1}(t),\ \ldots,\ x_{\alpha_1 \ldots \alpha_{k-1}}(t))$$

has nonzero partial derivatives,

$$\begin{aligned}\frac{\partial c}{\partial t^\beta}(t) &:= L_\beta(t,\ x(t),\ x_{\alpha_1}(t),\ \ldots,\ x_{\alpha_1 \ldots \alpha_k}(t)) \\ &= \frac{\partial S}{\partial t^\beta}(t,\ x(t),\ x_{\alpha_1}(t),\ \ldots, x_{\alpha_1 \ldots \alpha_{k-1}}(t)) \\ &+ \frac{\partial S}{\partial x^i}(t,\ x(t),\ x_{\alpha_1}(t),\ \ldots,\ x_{\alpha_1 \ldots \alpha_{k-1}}(t)) x^i_\beta(t) \\ &+ \sum_{\alpha_1 \ldots \alpha_r;\, r = \overline{1, k-1}} \frac{\partial S}{\partial x^i_{\alpha_1 \ldots \alpha_r}}(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1 \ldots \alpha_{k-1}}(t)) x^i_{\alpha_1 \ldots \alpha_r \beta}(t).\end{aligned} \tag{3.1}$$

For a fixed function $x(\cdot)$, let us define the *generalized multi-momenta* $p = (p^{\alpha_1 \ldots \alpha_j}_{\beta, i}),\ i \in \{1,\ 2,\ \ldots,\ n\},\ j \in \{1,\ 2,\ \ldots,\ k\},\ \alpha_j,\ \beta \in \{1,\ 2,\ \ldots,\ m\}$, by

$$p^{\alpha_1 \ldots \alpha_j}_{\beta, i}(t) := \frac{1}{n(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_j)} \frac{\partial L_\beta}{\partial x^i_{\alpha_1 \ldots \alpha_j}}(t,\ x(t),\ x_{\alpha_1}(t),\ \ldots,\ x_{\alpha_1 \ldots \alpha_k}(t)).$$

Remark 3.1. Here (for more details, the reader is directed to [9]),

$$n(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k) := \frac{|1_{\alpha_1} + 1_{\alpha_2} + \ldots + 1_{\alpha_k}|!}{(1_{\alpha_1} + 1_{\alpha_2} + \ldots + 1_{\alpha_k})!},$$

denotes the number of distinct indices represented by $\{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_k\}$, $\alpha_j \in \{1,\ 2,\ \ldots,\ m\},\ j = \overline{1,\ k}$.

By computation, for $j = \overline{1, k}$ and $\frac{\partial S}{\partial x^i_{\alpha_0}} := \frac{\partial S}{\partial x^i}$, we get the non-zero components of p, namely,

$$p^{\alpha_1 \dots \alpha_{j-1}\alpha_j}_{\alpha_j, i}(t) = \frac{1}{n(\alpha_1, \alpha_2, \dots, \alpha_j)} \frac{\partial S}{\partial x^i_{\alpha_1 \dots \alpha_{j-1}}}(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{k-1}}(t)).$$

Definition 3.1. The Lagrange 1-form $L_\beta(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_k}(t))$ is called *super-regular* if the algebraic system

$$p^{\alpha_1 \dots \alpha_j}_{\beta, i}(t) = \frac{1}{n(\alpha_1, \alpha_2, \dots, \alpha_j)} \frac{\partial L_\beta}{\partial x^i_{\alpha_1 \dots \alpha_j}}(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_k}(t)),$$

defines the function

$$x^i_{\alpha_1} = x^i_{\alpha_1}(t, x, p^{\alpha_1}_{\alpha_1}, \dots, p^{\alpha_1 \dots \alpha_k}_{\alpha_k}),$$

$$\vdots$$

$$x^i_{\alpha_1 \dots \alpha_k} = x^i_{\alpha_1 \dots \alpha_k}(t, x, p^{\alpha_1}_{\alpha_1}, \dots, p^{\alpha_1 \dots \alpha_k}_{\alpha_k}).$$

In these conditions, the previous relations become

$$x^i_{\alpha_1} = x^i_{\alpha_1}\left(t, x, \frac{\partial S}{\partial x}, \dots, \frac{1}{n(\alpha_1, \alpha_2, \dots, \alpha_k)} \frac{\partial S}{\partial x_{\alpha_1 \dots \alpha_{k-1}}}\right),$$

$$\vdots$$

$$x^i_{\alpha_1 \dots \alpha_k} = x^i_{\alpha_1 \dots \alpha_k}\left(t, x, \frac{\partial S}{\partial x}, \dots, \frac{1}{n(\alpha_1, \alpha_2, \dots, \alpha_k)} \frac{\partial S}{\partial x_{\alpha_1 \dots \alpha_{k-1}}}\right).$$

On the other hand, the relation (3.1) can be rewritten as

$$-\frac{\partial S}{\partial t^\beta}(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{k-1}}(t))$$

$$= \frac{\partial S}{\partial x^i}(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{k-1}}(t))$$

$$\cdot x^i_\beta\left(t,\ x,\ \frac{\partial S}{\partial x}(\cdot),\ \dots,\ \frac{1}{n(\alpha_1,\ \alpha_2,\ \dots,\ \alpha_k)}\frac{\partial S}{\partial x_{\alpha_1\dots\alpha_{k-1}}}(\cdot)\right)$$

$$+\sum_{\alpha_1\dots\alpha_r;\ r=\overline{1,\ k-1}}\frac{\partial S}{\partial x^i_{\alpha_1\dots\alpha_r}}(t,\ x(t),\ x_{\alpha_1}(t),\ \dots,\ x_{\alpha_1\dots\alpha_{k-1}}(t))$$

$$\cdot x^i_{\alpha_1\dots\alpha_r\beta}\left(t,\ x,\ \frac{\partial S}{\partial x}(\cdot),\ \dots,\ \frac{1}{n(\alpha_1,\ \alpha_2,\ \dots,\ \alpha_k)}\frac{\partial S}{\partial x_{\alpha_1\dots\alpha_{k-1}}}(\cdot)\right)$$

$$-\ L_\beta(t,\ x(t),\ x_{\alpha_1}(t),\ \dots,\ x_{\alpha_1\dots\alpha_k}(t)). \tag{3.2}$$

The super-regular Lagrange 1-form L_β enters in duality with the following Hamiltonian 1-form:

$$H_\beta(t,\ x,\ p^{\alpha_1}_{\alpha_1},\ \dots,\ p^{\alpha_1\dots\alpha_k}_{\alpha_k})$$

$$=\sum_{\alpha_1\dots\alpha_j;\ j=\overline{1,\ k}}\frac{1}{n(\alpha_1,\ \alpha_2,\ \dots,\ \alpha_j)}x^i_{\alpha_1\dots\alpha_j}(t,\ x,\ p^{\alpha_1}_{\alpha_1},\ \dots,\ p^{\alpha_1\dots\alpha_k}_{\alpha_k})$$

$$\cdot\frac{\partial L_\beta}{\partial x^i_{\alpha_1\dots\alpha_j}}(t,\ x,\ \dots,\ x_{\alpha_1\dots\alpha_k}(\cdot))$$

$$-\ L_\beta(t,\ x,\ x_{\alpha_1}(t,\ x,\ p^{\alpha_1}_{\alpha_1},\ \dots,\ p^{\alpha_1\dots\alpha_k}_{\alpha_k}),\ \dots,\ x_{\alpha_1\dots\alpha_k}$$

$$(t,\ x,\ p^{\alpha_1}_{\alpha_1},\ \dots,\ p^{\alpha_1\dots\alpha_k}_{\alpha_k})),$$

(*multi-time higher-order non-standard Legendrian duality*) or, shortly,

$$H_\beta = x^i_{\alpha_1\dots\alpha_j}p^{\alpha_1\dots\alpha_j}_{\beta,\ i} - L_\beta.$$

Now, we can rewrite (3.2) as *Hamilton-Jacobi system of PDEs based on higher-order Lagrangians*

$$\frac{\partial S}{\partial t^\beta}+H_\beta\left(t,\ x,\ \frac{\partial S}{\partial x},\ \dots,\ \frac{\partial S}{\partial x_{\alpha_1\dots\alpha_{k-1}}}\right)=0,\quad \beta\in\{1,\ \dots,\ m\}.$$

Usually, the Hamilton-Jacobi system of PDEs based on higher-order Lagrangians is accompanied by the initial condition

$$S(0, x, x_{\alpha_1}, \dots, x_{\alpha_1 \dots \alpha_{k-1}}) = S_0(x, x_{\alpha_1}, \dots, x_{\alpha_1 \dots \alpha_{k-1}}).$$

The solution $S(t, x, x_{\alpha_1}, \dots, x_{\alpha_1 \dots \alpha_{k-1}})$ is called the *generating function* of the generalized multi-momenta.

Remark 3.2. Conversely, let $S(t, x, x_{\alpha_1}, \dots, x_{\alpha_1 \dots \alpha_{k-1}})$ be a solution of the Hamilton-Jacobi system of PDEs based on higher-order Lagrangians. We assume (the non-zero components of p)

$$p_{\alpha_j, i}^{\alpha_1 \dots \alpha_{j-1} \alpha_j}(t) := \frac{1}{n(\alpha_1, \alpha_2, \dots, \alpha_j)} \frac{\partial S}{\partial x^i_{\alpha_1 \dots \alpha_{j-1}}}(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{k-1}}(t)),$$

for $j = \overline{1, k}$ and $\dfrac{\partial S}{\partial x^i_{\alpha_0}} := \dfrac{\partial S}{\partial x^i}$.

Then, the following formula shows that the action integral can be written as a path independent curvilinear integral:

$$\int_{\Gamma_{t_0, t_1}} L_\beta(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_k}(t)) dt^\beta$$

$$= \int_{\Gamma_{t_0, t_1}} \left[x^i_{\alpha_1 \dots \alpha_j}(t) p_{\beta, i}^{\alpha_1 \dots \alpha_j}(t) - H_\beta\left(t, x, \frac{\partial S}{\partial x}(\cdot), \dots, \frac{\partial S}{\partial x_{\alpha_1 \dots \alpha_{k-1}}}(\cdot)\right)\right] dt^\beta$$

$$= \int_\Gamma \sum_{\alpha_1 \dots \alpha_{j-1};\, j = \overline{1, k}} \frac{1}{n(\alpha_1, \alpha_2, \dots, \alpha_j)} \frac{\partial S}{\partial x^i_{\alpha_1 \dots \alpha_{j-1}}} dx^i_{\alpha_1 \dots \alpha_{j-1}} + \frac{\partial S}{\partial t^\beta} dt^\beta = \int_\Gamma dS.$$

Theorem 3.1. *The generating function of the generalized multi-momenta is solution of the Cauchy problem*

$$\frac{\partial S}{\partial t^\beta} + H_\beta\left(t, x, \frac{\partial S}{\partial x}, \dots, \frac{\partial S}{\partial x_{\alpha_1 \dots \alpha_{k-1}}}\right) = 0, \quad \beta \in \{1, \dots, m\},$$

$$S(0, x, x_{\alpha_1}, \dots, x_{\alpha_1 \dots \alpha_{k-1}}) = S_0(x, x_{\alpha_1}, \dots, x_{\alpha_1 \dots \alpha_{k-1}}).$$

Application. Suppose t is the time, $x = (x^i)$ is the vector of spatial coordinates, the function (operator) $H_1 = I$ is associated with the information as a measure of organization (synergy and purpose), the function (operator) $H_2 = H$ with the energy as a measure of movement, the function S^1 is the generating function for entropy, and S^2 is the generating function for action. A PDEs system of the type

$$\frac{\partial S^1}{\partial t} + H_1\left(t,\, x,\, \frac{\partial S^1}{\partial x},\, \frac{\partial S^2}{\partial x}\right) = 0, \quad \frac{\partial S^2}{\partial t} + H_2\left(t,\, x,\, \frac{\partial S^1}{\partial x},\, \frac{\partial S^2}{\partial x}\right) = 0$$

is called *physical control*. This kind of system can be written using the real vector function $S = (S^1, S^2) : R \times R^n \to R.$

4. Conclusion

In the present paper, using a non-standard Legendrian duality for single-time and multi-time higher-order Lagrangians, we have introduced Hamilton-Jacobi PDE and Hamilton-Jacobi system of PDEs. In this way, our results have extended, unified and improved several existing theorems in the current literature.

References

[1] F. Cardin and C. Viterbo, Commuting Hamiltonians and Hamilton-Jacobi multi-time equations, Duke Math. J. 144(2) (2008), 235-284.

[2] N. Y. Ibragimov, Integrating factors, adjoint equations and Lagrangians, J. Math. Anal. Appl. 318(2) (2006), 742-757.

[3] O. Krupkova, Hamiltonian field theory revisited: A geometric approach to regularity, Steps in Differential Geometry, Proceedings of the Colloquium on Differential Geometry, 25-30 July, 2000, Debrecen, Hungary.

[4] L. P. Lebedev and M. J. Cloud, The Calculus of Variations and Functional Analysis with Optimal Control and Applications in Mechanics, World Scientific, Series A, 12, 2003.

[5] R. Miron, The Geometry of Higher Order Lagrange Spaces, Applications to Mechanics and Physics, Kluwer, FTPH no. 82, 1997.

[6] M. Motta and F. Rampazzo, Nonsmooth multi-time Hamilton-Jacobi systems, Indiana Univ. Math. J. 55(5) (2006), 1573-1614.

[7] J. C. Rochet, The taxation principle and multitime Hamilton-Jacobi equations, J. Math. Econom. 14(2) (1985), 113-128.

[8] M. R. Roman, Higher Order Lagrange Spaces, Applications, PhD Thesis, University of Iassy, 2001.

[9] D. J. Saunders, The Geometry of Jet Bundles, Cambridge Univ. Press, 1989.

[10] S. Treanţă, On multi-time Hamilton-Jacobi theory via second order Lagrangians, U.P.B. Sci. Bull., Series A: Appl. Math. Phys. 76(3) (2014), 129-140.

[11] S. Treanţă and C. Vârsan, Linear higher order PDEs of Hamilton-Jacobi and parabolic type, Math. Reports 16(66)(2) (2014), 319-329.

[12] S. Treanţă, PDEs of Hamilton-Pfaff type via multi-time optimization problems, U.P.B. Sci. Bull., Series A: Appl. Math. Phys. 76(1) (2014), 163-168.

[13] P. N. V. Tu, Introductory Optimization Dynamics, Springer-Verlag, 1991.

[14] C. Udrişte, Dynamics induced by second-order objects, BSG Proc. 4, Geometry Balkan Press (2000), 161-168.

[15] C. Udrişte and I. Ţevy, Multi-time Euler-Lagrange-Hamilton theory, WSEAS Transactions on Mathematics 6(6) (2007), 701-709.

[16] C. Udrişte, L. Matei and I. Duca, Multi-time Hamilton-Jacobi theory, Proceedings of the 8-th WSEAS International Conference on Applied Computer and Applied Computational Science (2009), 509-513.

■