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ON SOME GEOMETRIC METHODS IN DIFFERENTIAL EQUATIONS

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Abstract

We give a survey of geometric methods used in papers and books by V. I. Arnold and V. V. Kozlov. They are methods of different normal forms, of different polyhedra, of small denominators and of asymptotic expansions.

1. Introduction

In paper (Khesin and Tabachnikov [44]), there was given a short description of main achievements of Arnold. Below in Sections 2-4, we give some additions to several sections of this paper. In Sections 5, 6, 8, and 9, we discuss two kinds of normal forms in papers by Arnold and by Kozlov.

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Logarithmic branching solutions to Painlevé equations is discussed in Section 7.

In this paper, all the formulas denoted as (n^*) refer to the formulas (n) in the cited papers.

2. On the Last Paragraph at Page 381 of Khesin et al. ([44]) Devoted to Small Divisors

Arnold's theorem on stability of the stationary point in the Hamiltonian system with two degrees of freedom in Arnold [1] had wrong formulation (see Bruno ([14], § 12, Section IVd)). Then Arnold added one more condition in his theorem Arnold [2], but its proof was wrong because it used the wrong statement (see Bruno [17, 18]). All mathematical world was agreed with my critics except Arnold. On the other hand, in the first proof of the same theorem by Moser [55] there was a similar mistake (see Bruno [14], § 12, Section IVe)). But in Siegel et al. [63], Moser corrected his proof after my critics, published in Bruno ([14], § 12, Section IVe).

Concerning the KAM theory. In 1974, I developed its generalization via normal forms: Bruno ([15, 16, 20], Part II). But up-to-day almost nobody understands my generalization. Some connections with my approach see in the book Broer et al. [11].

3. On the Last Paragraph at page 384 of Khesin et al. ([44]) Concerning Higher-Dimensional Analogue of Continued Fraction

The paper Lauchand [53] "Polyèdre d'Arnol'd et ..." by Lauchand was presented to C. R. Acad. Sci. Paris by Arnold in 1993. See also preprint Lauchand [54]. When I saw the article I published the paper Bruno et al. [22] "Klein polyhedrals ..." (1994), because so-called "Arnold polyhedra" were proposed by Klein [45, 46] one hundred years early. Moreover, they were introduced by Skubenko [62] as well. In 1994-2000, I and Parusnikov ([56, 57, 59]) studied Klein polyhedra from algorithmical view point and found that they do not give a basis for algorithm generalizing the continued fraction. So in 2003, I proposed another approach and another sole polyhedron, which give a basis for the generalization in 3 and any dimensions (see Bruno [26, 27]; Bruno et al. [30]; Bruno [31, 32, 38, 39, 40]; Parusnikov [58]). Now, there are a lot of publications on the Klein polyhedra and their authors following after Arnold [9] wrongly think that the publications are on generalization of the continued fraction.

4. On the Last two Paragraphs at page 395 of Khesin et al. ([44]) Devoted to the Newton Polygon

In that text the term “Newton polygon” must be replaced by “Newton polyhedron”. In contemporary terms, Newton introduced *support* and one *extreme edge* of the *Newton open polygon* for one polynomial of two variables. The full Newton open polygon was proposed by Puiseux [61] and by Briot and Bouquet [10] for one ordinary differential equation of the first order. Firstly, a polyhedron as the convex hull of the support was introduced in my paper Bruno [12] for an autonomous system of n ODEs. During 1960-1969, Arnold wrote 3 reviews on my works devoted to polygons and polyhedrons for ODEs with sharp critics “of the geometry of power exponents” (see my book Bruno ([23], Chapter 8, Section 6)). Later in 1974, he introduced the name “Newton polyhedron” (see Gindikin [43]), made the view that it is his invention and never gave references on my work. Now I have developed “Universal Nonlinear Analysis” which allows to compute asymptotic expansions of solutions to equations of any kind (algebraic, ordinary differential and partial differential): Bruno [36].

5. On Non-Hamiltonian Normal Form

In my paper Bruno [13] and my candidate thesis “Normal form of differential equations” in 1966, I introduced normal forms in the form of power series. It was a new class of them. Known before normal forms were either linear (Poincare [60]) or polynomial (Dulac [41]). An official opponent to my thesis was Kolmogorov. He estimated very high that new class of normal forms. Arnold put my normal form into his books ([5, 8], § 23) without reference on my publication and named it as “Poincare-Dulac normal form”. So, readers of his book attributed my normal form to Arnold. I saw several articles where my normal form was named as Arnold’s.

6. On Canonical Normalizing Transformation

In Arnold et al. ([6, 7], Chapter 7, § 3, Subsection 3.1) a proof of Theorem 7 is based on construction of a generating function $F = \langle P, q\rangle + S_l(P, q)$ in mixed coordinates P, q. Transformation from old coordinates P, Q to new coordinates p, q is given by the formulae

$$p = \frac{\partial F}{\partial q}, \quad Q = \frac{\partial F}{\partial P}. \tag{1}$$

Here $S_l(P, q)$ is a homogeneous polynomial in P and q of order l. According to (1), the transformation from coordinates P, Q to coordinates p, q is given by infinite series, which are results of the resolution of the implicit equations (1). Thus, the next to the last sentence on page 272 (in Russian edition) “The normalizing transformation is constructed in the polynomial form of order $L-1$ in phase variables” is wrong. Indeed that property has the normalizing transformation computed by the Zhuravlev-Petrov method: Bruno et al. [28]; Zhuravlev et al. [64].

7. On Branching Solutions of Painlevé Equations

In Kozlov et al. ([50], Chapter I, § 4, Example 1.4.6), the Painlevé equations are successive considered. In particularly, there was find the expansion

$$x(\tau) = \tau^{-1}\sum_{k=0}^{\infty} x_k \tau^k, \tag{2}$$

of a solution to the fifth Painlevé equation. The series (2) is considered near the point $\tau = 0$. After the substitution $\tau = \log t$, we obtain the series

$$x(t) = \log^{-1} t \sum_{k=0}^{\infty} x_k \log^k t, \tag{3}$$

which has a sense near the point $t = 1$, where log $t = 0$. However, from the last expansion (3), authors concluded that $t = 0$ is the point of the logarithmic branching the solution $x(t)$. It is wrong, because the expansion (3) does not work for $t = 0$ as $\log 0 = \infty$ and the expansion (3) diverges. That mistake is in the first edition of the book Kozlov et al. [50] and was pointed out in the paper Bruno et al. [24], but it was not corrected in the second "corrected" edition of the book by Kozlov et al. [51] and in its English translation Kozlov et al. [52].

A similar mistake is there in consideration of the sixth Painlevé equation. There for a solution to the sixth Painlevé equation, it was obtained the expansion (2). After the substitution $\tau = \log(t(t-1))$, it takes the form

$$x(t) = \log^{-1}(t(t-1)) \sum_{k=0}^{\infty} x_k \log^k (t(t-1)).$$

As the expansion (2) has a sense near the point $\tau = 0$, the last expansion has a sense near points $t = (1 \pm \sqrt{5}\,)/2$, because in them $t(t-1) = 1$ and $\tau = 0$. Thus, the conclusion in the book, that points $t = 0$ and $t = 1$ are the logarithmic branching points of the solution, is non correct. The mistake was point out in the paper of Bruno et al. [25], but it was repeated in the second edition of the book by Kozlov et al. [51] and its English translation Kozlov et al. [52]. Indeed solutions of the Painlevé equations have logarithmic branching, see Bruno et al. [33, 34]; Bruno [37].

8. On Integrability of the Euler-Poisson Equations

In the paper by Kozlov [47], Theorem 1 on nonexistence of an additional analytic integral was applied in § 3 to the problem of motion of a rigid body around a fixed point. The problem was reduced to a Hamiltonian system with two degrees of freedom and with two parameters x, y. The system has a stationary point for all values of parameters. Condition on existence of the resonance 3 : 1 was written as Equation (6^*) on parameters x, y. Then, the second order form of the Hamiltonian function was reduced to the simplest form by a linear canonical transformation

$$(x_1, x_2, y_1, y_2) \to (q_1, q_2, p_1, p_2). \tag{4}$$

Condition of vanishing the resonant term of the fourth order in the obtained Hamiltonian function was written as Equation (7^*) on x, y. System of Equations (6^*) and (7^*) was considered for

$$x > 0 \quad \text{and} \quad y > \frac{x}{x+1},$$

where the system has two roots

$$x = \frac{4}{3},\ y = 1 \text{ and } x = 7,\ y = 2. \tag{5}$$

They correspond to two integrable cases $y = 1$ and $y = 2$ of the initial problem. It was mentioned in Theorem 3. But in the whole real plane (x, y) the system of Equations (6^*) and (7^*) has roots (5) and three additional roots

$$x = -\frac{16}{3},\ y = 1; \qquad x = -\frac{17}{9},\ y = 2; \tag{6}$$

$$x = 0,\ y = 9. \tag{7}$$

Roots (6) belong to integrable cases $y = 1$ and $y = 2$. But the root (7) is out of them. Indeed the transformation (4) is not defined for $x = 0$. If to make an additional analysis for $x = 0$, then for resonance 3 : 1 one obtains two points: (7) and

$$x = 0, \quad y = \frac{1}{9}. \tag{8}$$

In both these points, the resonant term of the fourth order part of the Hamiltonian function vanishes. But points (7) and (8) are out of the integrable cases $y = 1$ and $y = 2$; they contradict to statement of Theorem 3 Kozlov [47]. The paper Kozlov [47] was repeated in the books Kozlov ([48, 49], Chapter VI, § 3, Section 3). A non-Hamiltonian study of the problem see in the paper Bruno ([29], Section 5). Nonintegrability at the points (7) and (8) was shown in Bruno [35].

9. On Normal forms of Families of Linear Hamiltonian Systems

Real normal forms of families of linear Hamiltonian systems were given in Galin ([42], § 2), where formula (16^*) wrongly indicated the normal form corresponding to the elementary divisor λ^{2l}: the third sum in the formula (16^*) has to be omitted. The indicated mistake was reproduced in the first three Russian editions of the book Arnold ([3], Appendix 6) and in its English translation Arnold ([4], Appendix 6). Discussions of that see in the paper Bruno [19] and in the book Bruno ([21], Chapter I, Section 6, Notes to Subsection 1.3).

10. Conclusion

Sections 2-4 were sent to Notices of the AMS for publication as a letter to the editor. But Editor S. G. Krantz rejected it. I consider that as one more case of the scientific censorship in the AMS.

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