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R_4 OF CLASS ONE

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Abstract

For spacetimes embedded into flat 5-space, with Lanczos invariant different to zero, we obtain the inverse of the corresponding second fundamental form.

1. Introduction

R_4 can be embedded into E_5 (that is, the 4-space has class one) if and only if, there exist the second fundamental form $b_{ac} = b_{ca}$ satisfying the Gauss-Codazzi equations [1]

$$R_{acij} = \epsilon\,(b_{ai}b_{cj} - b_{aj}b_{ci}), \tag{1}$$

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$$b_{ij;\,c} = b_{ic;\,j}, \tag{2}$$

where $\epsilon = \pm 1$, R_{acij} is the Riemann tensor; and r means covariant derivative. It is well-known [2] that whenever $\det(b^i{}_j) \neq 0$, then (1) implies (2), in other words, when a nonsingular matrix $\widetilde{b}$ satisfies the Gauss equation, then the Codazzi equation is verified automatically. However, in general, the construction of $\widetilde{b}$ for a given spacetime should involve the study of both (1) and (2) together.

In [3] were obtained the following results:

$$-24\det(b^i{}_j) = K_2 \equiv \; {}^*R^{*ijac}R_{acij}, \tag{3}$$

$$pb_{ic} = \frac{1}{48}K_2 g_{ic} - \frac{1}{2}R_{ijrc}G^{jr}, \qquad p = \frac{\epsilon}{3}b^{ac}G_{ac}, \tag{4}$$

such that K_2 is a Lanczos invariant [4-6] defined in terms of double dual of curvature tensor

$${}^*R^{*ij}{}_{ac} = \frac{1}{4}\eta^{ijrt}R_{rt}{}^{mn}\eta_{mnac}, \tag{5}$$

with η_{ijrt} denoting the Levi-Civita symbol, and G_{ac} is the Einstein tensor [1]. Besides, the Bianchi identities [1] adopt the compact form [5]

$${}^*R^{*rjic}{}_{;i} = 0. \tag{6}$$

The present work deals with the case $K_2 \neq 0$, thus in according to (3), this implies the existence of inverse matrix $b^{-1}{}_{ij}$. In the next section, $\widetilde{b}^{-1}$ is constructed as a projection of $\widetilde{b}$ onto double dual tensor.

2. Inverse of Second Fundamental Form

If (1) is employed in the Lanczos identity [4, 7]

$$K_2 g_{ra} = 4\,{}^*R^*{}_{rjqc}R_a{}^{jqc}, \tag{7}$$

it is immediate that

$$K_2 g_{ra} = 8\epsilon\, {}^*R^*{}_{rjqc} b_a{}^q b^{jc},$$

and its multiplication by b^{-1ia} implies

$$K_2 b^{-1}{}_{ri} = 8\epsilon\, {}^*R^*{}_{rjic} b^{jc}, \tag{8}$$

thus $\widetilde{b}^{-1}$ essentially is the projection of $\widetilde{b}$ onto (5). It is interesting to observe that (2), (6), and (8) generate the differential condition

$$(K_2 b^{-1ri})_{;i} = 0, \tag{9}$$

that is, (8) is a conserved tensor [8] for any 4-space of class one with $K_2 \neq 0$.

The relations (1) and (4) permit to deduce an alternative expression for $\widetilde{b}^{-1}$, in fact, if the Gauss equation is applied into (4)

$$p b_{ic} = -\frac{1}{24} K_2 g_{ic} + \epsilon b_{ir} b_{jc} G^{rj},$$

whose multiplication by $b^{-1c}{}_a$ leads to

$$\frac{1}{24} K_2 b^{-1}{}_{ia} = \epsilon b_{ir} G^r{}_a - p g_{ia}, \tag{10}$$

in harmony with (8).

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