Research and Communications in Mathematics and Mathematical Sciences
Vol. 5, Issue 1, 2015, Pages 61-67
ISSN 2319-6939
Published Online on May 11, 2015

http://jyotiacademicpress.net

ORBITS OF FINITE SETS UNDER SYMMETRIC GROUPS

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Abstract

Let n be a positive integer and $[n] := \{1, 2, \dots, n\}$. Let S_n be the symmetric group on $[n]$. This article describes the orbits of $[n]^t$ under S_n, computes the number of the orbits and the length of each orbit, where $[n]^t := \underbrace{[n] \times [n] \times \cdots \times [n]}_{t}$.

1. Introduction

Let G be a group and X be a set, if there is a function $G \times X \to X$ (usually denoted by $(g, x) \to gx$) such that for all $x \in X$ and $g_1, g_2 \in G$:

$$ex = x, \ (g_1 g_2)x = g_1(g_2 x),$$

then we say that the group G acts on the set X, where e is the identity element of the group G.

2010 Mathematics Subject Classification: 20G20, 05E15.
Keywords and phrases: symmetric group, orbit, length of orbit.
Communicated by Bao-Xuan Zhu.
Received April 24, 2015

Let G be a group that acts on a set X, the relation on X is defined by

$$x \sim y \Leftrightarrow gx = y \text{ for some } g \in G.$$

It is well-known that the relation is an equivalence relation. The equivalence classes of the above equivalence relation are called the orbits of X under G. For $x \in X$, the orbit of x is the set

$$O_x = \{gx | g \in G\}.$$

For $x \in X$, the subset

$$H_x = \{g \in G | gx = x\},$$

is a subgroup of G. H_x is called the isotropy group of x.

An action of the group G on the set X is said to be *transitive*, if there is $g \in G$ such that $y = gx$, for all $x, y \in X$.

Let n be a positive integer and $[n] := \{1, 2, \ldots, n\}$. Let S_n be the symmetric group on $[n]$. There is an action of the symmetric group S_n on $[n]$ defined as follows

$$\begin{array}{ccc} [n] \times S_n & \rightarrow & [n] \\ (i, \sigma) & \mapsto & \sigma(i). \end{array}$$

It is well-known that S_n is transitive on $[n]$.

Let t be a positive integer and $[n]^t := \underbrace{[n] \times [n] \times \cdots \times [n]}_{t}$. Then we can get the following natural action of S_n on $[n]^t$,

$$\begin{array}{ccc} [n]^t \times S_n & \rightarrow & [n]^t \\ ((i_1, i_2, \ldots, i_t), \sigma) & \mapsto & (\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_t)). \end{array}$$

If $t \geq 2$, then S_n is not transitive on $[n]^t$ in general. This article describes the orbits of $[n]^t$ under S_n, computes the number of the orbits and the length of each orbit.

Guo et al. [1-7] studied the orbits of subspaces under classical groups, which are subgroups of symmetric groups.

2. Main Results

In this section, we begin with a useful lemma.

Lemma 2.1 ([8]). *Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, \ldots, n_k$, respectively. Let the size of S be $A = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of S equals*

$$\frac{A!}{n_1!\, n_2! \cdots !\, n_k!}.$$

Let $(i_1, i_2, \ldots, i_t) \in [n]^t$. If there are exactly s different elements in $i_1, i_2, \ldots, i_t$, then $(i_1, i_2, \ldots, i_t)$ is called a t-repetitive permutation of size s. The set of all t-repetitive permutations of size s is denoted by $[n]_s^t$ with $1 \leq s \leq t$. For $(i_1, i_2, \ldots, i_t) \in [n]_s^t$, let $i_{k_1}, i_{k_2}, \ldots, i_{k_s}$ be the s different elements in $i_1, i_2, \ldots, i_t$. Assume that i_{k_r} appears m_r times in $(i_1, i_2, \ldots, i_t)$, where $1 \leq r \leq s$. If there are exactly q different elements in $m_1, m_2, \ldots, m_s$, and they appear $l_1, l_2, \ldots, l_q$ times in $(m_1, m_2, \ldots, m_s)$, respectively, then we define $\Theta(m_1, m_2, \ldots, m_s) := l_1!\, l_2! \ldots l_q!$. By Lemma 2.1, we can obtain the following result.

Lemma 2.2. *Let s and t be positive integers with $1 \leq s \leq t$. Then*

$$|[n]_s^t| = \sum_{\substack{m_1 + m_2 + \cdots + m_s = t \\ m_r \geq 1 (1 \leq r \leq s)}} \frac{t!}{m_1!\, m_2! \cdots !\, m_s!}.$$

Theorem 2.3. *Let s and t be positive integers with $1 \leq s \leq t$. Then the number of the orbits of $[n]_s^t$ under S_n is*

$$\sum_{\substack{m_1 + m_2 + \cdots + m_s = t \\ m_1 \geq m_2 \geq \cdots \geq m_s \geq 1}} \frac{t!}{m_1!\, m_2! \cdots m_s!\, \Theta(m_1, m_2, \ldots, m_s)},$$

and the length of each orbit is $n(n-1) \cdots (n-s)$.

Proof. Let $i_{k_1}, i_{k_2}, \ldots, i_{k_s}$ be s different elements in $i_1, i_2, \ldots, i_t$, and i_{k_r} appears m_r times in $(i_1, i_2, \ldots, i_t)$ with $1 \leq r \leq s$. For any $(i_1, i_2, \ldots, i_t), (j_1, j_2, \ldots, j_t) \in [n]_s^t$, they are in the same orbit under S_n if and only if there exists $\sigma \in S_n$ such that

$$(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_t)) = (j_1, j_2, \ldots, j_t).$$

By the transitivity of S_n on $[n]$ and the definition of $\Theta(m_1, m_2, \ldots, m_s)$, the number of the orbits of $[n]_s^t$ under S_n is

$$\sum_{\substack{m_1+m_2+\cdots+m_s=t \\ m_1 \geq m_2 \geq \cdots \geq m_s \geq 1}} \frac{t!}{m_1!\, m_2! \cdots m_s!\, \Theta(m_1, m_2, \ldots, m_s)}.$$

It is easy to see that the length of each orbit is $n(n-1)\cdots(n-s)$.

Corollary 2.4. *Let $n \geq t$. Then the number of the orbits of $[n]^t$ under S_n is*

$$\sum_{s=1}^{t} \sum_{\substack{m_1+m_2+\cdots+m_s=t \\ m_1 \geq m_2 \geq \cdots \geq m_s \geq 1}} \frac{t!}{m_1!\, m_2! \cdots m_s!\, \Theta(m_1, m_2, \ldots, m_s)}.$$

Corollary 2.5. *Let $n < t$. Then the number of the orbits of $[n]^t$ under S_n is*

$$\sum_{s=1}^{n} \sum_{\substack{m_1+m_2+\cdots+m_s=t \\ m_1 \geq m_2 \geq \cdots \geq m_s \geq 1}} \frac{t!}{m_1!\, m_2! \cdots m_s!\, \Theta(m_1, m_2, \ldots, m_s)}.$$

3. Examples

In this section, we give the orbits of $[n]^t$ under S_n for $t = 2, 3, 4$ in detail.

Example 3.1. If $n \geq 2$, the orbits of $[n]^2$ under S_n are

$$R_0 = \{(\sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_n\}, \; R_1 = \{(\sigma(1), \sigma(2)) | \text{ for all } \sigma \in S_n\},$$

and the lengths of the orbits are

$$|R_0| = n, |R_1| = n^2 - n.$$

Example 3.2. If $n \geq 3$, the orbits of $[n]^3$ under S_n are

$$R_0 = \{(\sigma(1), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_1 = \{(\sigma(1), \sigma(1), \sigma(2)) | \text{ for all } \sigma \in S_n\},$$

$$R_2 = \{(\sigma(1), \sigma(2), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_3 = \{(\sigma(2), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_4 = \{(\sigma(1), \sigma(2), \sigma(3)) | \text{ for all } \sigma \in S_n\},$$

and the lengths of the orbits are

$$|R_0| = n, |R_1| = |R_2| = |R_3| = n(n-1), |R_4| = n(n-1)(n-2).$$

If $n = 2$, the orbits of $[2]^3$ under S_2 are

$$R_0 = \{(\sigma(1), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_2\},$$

$$R_1 = \{(\sigma(1), \sigma(1), \sigma(2)) | \text{ for all } \sigma \in S_2\},$$

$$R_2 = \{(\sigma(1), \sigma(2), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_3 = \{(\sigma(2), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_2\}.$$

Example 3.3. If $n \geq 4$, the orbits of $[n]^4$ under S_n are

$$R_0 = \{(\sigma(1), \sigma(1), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_1 = \{(\sigma(1), \sigma(1), \sigma(2), \sigma(2)) | \text{ for all } \sigma \in S_n\},$$

$$R_2 = \{(\sigma(1), \sigma(2), \sigma(1), \sigma(2)) | \text{ for all } \sigma \in S_n\},$$

$$R_3 = \{(\sigma(1), \sigma(2), \sigma(2), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_4 = \{(\sigma(1), \sigma(1), \sigma(2), \sigma(3)) | \text{ for all } \sigma \in S_n\},$$

$$R_5 = \{(\sigma(1), \sigma(2), \sigma(1), \sigma(3)) | \text{ for all } \sigma \in S_n\},$$

$$R_6 = \{(\sigma(1), \sigma(2), \sigma(3), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_7 = \{(\sigma(2), \sigma(1), \sigma(3), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_8 = \{(\sigma(2), \sigma(3), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_9 = \{(\sigma(2), \sigma(1), \sigma(1), \sigma(3)) | \text{ for all } \sigma \in S_n\},$$

$$R_{10} = \{(\sigma(1), \sigma(1), \sigma(1), \sigma(2)) | \text{ for all } \sigma \in S_n\},$$

$$R_{11} = \{(\sigma(1), \sigma(1), \sigma(2), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_{12} = \{(\sigma(1), \sigma(2), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_{13} = \{(\sigma(2), \sigma(1), \sigma(1), \sigma(1)) | \text{ for all } \sigma \in S_n\},$$

$$R_{14} = \{(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) | \text{ for all } \sigma \in S_n\},$$

and the lengths of the orbits are

$$|R_0| = n, |R_1| = |R_2| = |R_3| = |R_{10}| = |R_{11}| = |R_{12}| = |R_{13}| = n(n-1),$$

$$|R_4| = |R_5| = |R_6| = |R_7| = |R_8| = |R_9| = n(n-1)(n-2),$$

$$|R_{14}| = n(n-1)(n-2)(n-3).$$

If $n = 3$, the orbits of $[3]^4$ under S_3 are $R_0, R_1, R_2, \dots, R_{13}$.

If $n = 2$, the orbits of $[2]^4$ under S_2 are $R_0, R_1, R_2, R_3, R_{10}, R_{11}, R_{12}, R_{13}$.

Acknowledgement

This research is supported by the Foundation of Langfang Teachers University (LSZQ201003) and Natural Science Foundation of Hebei Education Department (YQ2014018).

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