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INEQUALITIES ON THE LINE SEGMENT

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Abstract

The article deals with generalizations of the inequalities for convex functions on the line segment. The Jensen and the Hermite-Hadamard inequalities are included in the study. Some improvements of the Hermite-Hadamard inequality are obtained and applied to mathematical means.

1. Introduction

Let $\mathbb{X}$ be a real linear space. A linear combination $\alpha a + \beta b$ of points $a, b \in \mathbb{X}$ and coefficients $\alpha, \beta \in \mathbb{R}$ is affine if $\alpha + \beta = 1$. A set $\mathcal{S} \subseteq \mathbb{X}$ is affine if it contains all binomial affine combinations of its points. A function $h : \mathcal{S} \to \mathbb{R}$ is affine if the equality

$$h(\alpha a + \beta b) = \alpha h(a) + \beta h(b), \tag{1}$$

holds for every binomial affine combination $\alpha a + \beta b$ of the affine set $\mathcal{S}$.

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Convex combinations and sets are introduced by restricting to affine combinations with nonnegative coefficients. A function $h : \mathcal{S} \to \mathbb{R}$ is convex if the inequality

$$f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b), \tag{2}$$

holds for every binomial convex combination $\alpha a + \beta b$ of the convex set $\mathcal{S}$.

Using mathematical induction, the above concept can be extended to n-membered affine or convex combinations.

In this paper, we use the real line $\mathbb{X} = \mathbb{R}$. Besides convex and affine combinations, we will use barycenters of the sets of real numbers. If μ is a positive measure on $\mathbb{R}$, and if $\mathcal{S} \subseteq \mathbb{R}$ is a measurable set such that $\mu(\mathcal{S}) > 0$, then the integral mean point

$$c = \frac{1}{\mu(\mathcal{S})} \int_{\mathcal{S}} x \, d\mu, \tag{3}$$

is called the barycenter of the set $\mathcal{S}$ respecting the measure μ, or just the set barycenter. The barycenter c belongs to the convex hull of the set $\mathcal{S}$, as the smallest convex set containing $\mathcal{S}$. Given the measurable set $\mathcal{S}$ of positive measure, every affine function $h : \mathbb{R} \to \mathbb{R}$ satisfies the equality

$$h\left(\frac{1}{\mu(\mathcal{S})} \int_{\mathcal{S}} x \, d\mu\right) = \frac{1}{\mu(\mathcal{S})} \int_{\mathcal{S}} h(x) d\mu. \tag{4}$$

For the purpose of the paper, the set $\mathcal{S}$ will be used as an interval or a union of intervals.

2. The Jensen and the Hermite-Hadamard Inequalities

Through the paper, we will use a bounded interval of real numbers with endpoints $a < b$. Each point $c \in [a, b]$ can be presented by the unique binomial convex combination

$$c = \alpha a + \beta b, \tag{5}$$

where

$$\alpha = \frac{b-c}{b-a}, \quad \beta = \frac{c-a}{b-a}. \tag{6}$$

The next two lemmas present the properties of a convex function $f : [a, b] \to \mathbb{R}$ concerning its supporting and secant line.

The discrete version refers to interval points and interval endpoints sharing the common center.

Lemma A. *Let* $[a, b]$ *be a closed interval of real numbers, and let* $\sum_{i=1}^{n} \lambda_i x_i$ *be a convex combination of points* $x_i \in [a, b]$. *Let* $\alpha a + \beta b$ *be the unique endpoints convex combination such that*

$$\sum_{i=1}^{n} \lambda_i x_i = \alpha a + \beta b. \tag{7}$$

Then every convex function $f : [a, b] \to \mathbb{R}$ *satisfies the double inequality*

$$f(\alpha a + \beta b) \le \sum_{i=1}^{n} \lambda_i f(x_i) \le \alpha f(a) + \beta f(b). \tag{8}$$

Proof. Taking $c = \sum_{i=1}^{n} \lambda_i x_i$, we have the following two cases.

If $c \in \{a, b\}$, then Equation (8) is reduced to $f(c) \le f(c) \le f(c)$.

If $c \in (a, b)$, then using a supporting line $y = h_1(x)$ of the convex curve $y = f(x)$ at the graph point $C(c, f(c))$, and the secant line $y = h_2(x)$ passing through the graph points $A(a, f(a))$ and $B(b, f(b))$, we get the inequality

$$f(\alpha a + \beta b) = h_1(\alpha a + \beta b) = \sum_{i=1}^{n} \lambda_i h_1(x_i)$$

$$\leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

$$\leq \sum_{i=1}^{n} \lambda_i h_2(x_i) = h_2(\alpha a + \beta b) = \alpha f(a) + \beta f(b), \tag{9}$$

containing Equation (8). □

The discrete-integral version refers to the connection of the interval barycenter with interval endpoints.

Lemma B. *Let* $[a, b]$ *be a closed interval of real numbers, and let* μ *be a positive measure on* $\mathbb{R}$ *such that* $\mu([a, b]) > 0$. *Let* $\alpha a + \beta b$ *be the unique endpoints convex combination such that*

$$\frac{1}{\mu([a, b])} \int_{[a,b]} x \, d\mu = \alpha a + \beta b. \tag{10}$$

Then every convex function $f : [a, b] \to \mathbb{R}$ *satisfies the double inequality*

$$f(\alpha a + \beta b) \leq \frac{1}{\mu([a, b])} \int_{[a,b]} f(x) d\mu \leq \alpha f(a) + \beta f(b). \tag{11}$$

Proof. The proof can be done utilizing Equation (9) so that the integral means are used instead of the n-membered convex combinations. □

We emphasize the basic content of Lemma A. Using the left-hand side of the inequality in Equation (8) with the n-membered convex combination instead of the binomial endpoints convex combination, we obtain the discrete form of Jensen's inequality

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i). \tag{12}$$

Using the Riemann integral in Lemma B, the condition in (10) gives the midpoint

$$\frac{1}{b-a}\int_a^b x\,dx = \frac{1}{2}a + \frac{1}{2}b, \tag{13}$$

and its use in Equation (11) implies the classic Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{14}$$

Moreover, the inequality in Equation (14) follows by integrating the supporting-secant line inequality

$$h_1(x) \le f(x) \le h_2(x), \tag{15}$$

over the interval $[a, b]$.

We finish the section with a historic note on these two important inequalities. In 1905, applying the inductive principle, Jensen (see [4]) extended the inequality in Equation (2) to n-membered convex combinations. In 1906, working on transition to integrals, Jensen (see [5]) stated the another form. In 1883, studying convex functions, Hermite (see [3]) attained the inequality in Equation (14). In 1893, not knowing Hermite's result, Hadamard (see [2]) got the left-hand side of Equation (14). For information as regards the Jensen and the Hermite-Hadamard inequalities, one may refer to papers [1], [6], [9], [10], [11], and [12].

3. Main Results

To refine the Hermite-Hadamard inequality in Equation (14), we will use convex combinations of points of the closed interval $[a, b]$. In the main Theorem 3.1, we improve Equation (14) by using convex combinations of the midpoint $(a+b)/2$. The concluding Theorem 3.4 presents the integral refinement of Equation (14).

We take points $c, d \in [a, b]$ such that

$$\frac{c+d}{2} = \frac{a+b}{2}. \tag{16}$$

Applying the right-hand side of the inequality in Equation (8) to the above assumption, and multiplying by 2, we obtain the simple inequality

$$f(c) + f(d) \leq f(a) + f(b), \tag{17}$$

that will be used in this section. The main theorem follows.

Theorem 3.1. *Let* $[a, b]$ *be a closed interval of real numbers, and let* $[c, d] \subset (a, b)$ *be a closed subinterval satisfying the common barycenter condition in Equation* (16).

Then every convex function $f : [a, b] \to \mathbb{R}$ *satisfies the series of inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{4} f\left(\frac{a+c}{2}\right) + \frac{1}{2} f\left(\frac{c+d}{2}\right) + \frac{1}{4} f\left(\frac{d+b}{2}\right) \\ &\leq \frac{1}{4} \frac{\int_a^c f(x)dx}{c-a} + \frac{1}{2} \frac{\int_c^d f(x)dx}{d-c} + \frac{1}{4} \frac{\int_d^b f(x)dx}{b-d} \\ &\leq \frac{f(a) + 3f(c) + 3f(d) + f(b)}{8} \leq \frac{f(a) + f(b)}{2}. \end{aligned} \tag{18}$$

Proof. Applying the Hermite-Hadamard inequality to the convex combination of points c and d written as

$$t = \frac{1}{2} c + \frac{1}{2} d, \tag{19}$$

we have

$$\begin{aligned} f(t) &= f\left(\frac{c+d}{2}\right) \\ &\leq \frac{\int_c^d f(x)dx}{d-c} \\ &\leq \frac{f(c) + f(d)}{2}. \end{aligned} \tag{20}$$

Applying the same procedure to the convex combination of midpoints $(a+c)/2$ and $(d+b)/2$ given as

$$t = \frac{1}{2}\frac{a+c}{2} + \frac{1}{2}\frac{d+b}{2}, \tag{21}$$

we get

$$\begin{aligned} f(t) &\le \frac{1}{2} f\left(\frac{a+c}{2}\right) + \frac{1}{2} f\left(\frac{d+b}{2}\right) \\ &\le \frac{1}{2}\frac{\int_a^c f(x)dx}{c-a} + \frac{1}{2}\frac{\int_d^b f(x)dx}{b-d} \\ &\le \frac{f(a)+f(c)+f(d)+f(b)}{4}. \end{aligned} \tag{22}$$

Taking the arithmetic means of the inequalities in Equations (20) and (22), using Equation (17) and rearranging, we obtain the inequality in Equation (18). □

The inequality in Equation (18) can be expressed using the point $d = a + b - c$. The observed Equation (18) can also be expressed with the point $c = a + 2\delta$, where $0 < \delta < (b-a)/2$. Using this choice, we have $d = b - 2\delta$, $(a+c)/2 = a+\delta$ and $(d+b)/2 = b-\delta$. Finally, we can use the convex combinations

$$c = \alpha a + \beta b,\ d = (1-\alpha)a + (1-\beta)b, \tag{23}$$

provided that $\alpha a + \beta b < (a+b)/2$. Regardless of all these cases, the midpoint $(a+b)/2$ is not covered on the right side of Equation (18).

The inequality in Equation (18) does not include the case $c = d = (a+b)/2$. A method similar to that in Theorem 3.1 can be applied to intervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$ and so derive the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right]$$

$$\leq \frac{1}{b-a}\int_a^b f(x)dx$$

$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \qquad (24)$$

The above improvement of the Hermite-Hadamard inequality was noted in [8].

A convex function $f : [a, b] \to \mathbb{R}$ satisfies the inequality

$$f(a+b-c) \leq f(a)+f(b)-f(c), \qquad (25)$$

for every point $c \in [a, b]$ by Equation (17). The above simple inequality can be generalized by using the convex combination $\sum_{i=1}^{n}\gamma_i c_i$ instead of the point *c*. Applying Jensen's inequality to the convex combination

$$t = a+b-\sum_{i=1}^{n}\gamma_i c_i = \sum_{i=1}^{n}\gamma_i(a+b-c_i), \qquad (26)$$

and using Equation (25), Mercer (see [7]) obtained the inequality

$$f\left(a+b-\sum_{i=1}^{n}\gamma_i c_i\right) \leq f(a)+f(b)-\sum_{i=1}^{n}\gamma_i f(c_i). \qquad (27)$$

Corollary 3.2. *Let* $[a, b]$ *be a closed interval of real numbers, let* $c \in (a, b)$ *be an open interval point, and let* $f : [a, b] \to \mathbb{R}$ *be a convex function.*

If $c \leq (a+b)/2$, *then*

$$f(a+b-c)+f(c) \leq \frac{\int_a^c f(x)dx + \int_{a+b-c}^{c} f(x)dx}{c-a} \leq f(a)+f(b). \qquad (28)$$

If $c \geq (a+b)/2$, *then*

$$f(a+b-c)+f(c) \leq \frac{\int_a^{a+b-c} f(x)dx + \int_c^b f(x)dx}{c-a} \leq f(a)+f(b). \qquad (29)$$

Proof. We prove the double inequality in Equation (28). Let $y = h(x)$ be the secant line of the convex curve $y = f(x)$ passing through the graph points $C(c, f(c))$ and $D(a + b - c, f(a + b - c))$. Using the affinity of the secant function h specified in Equations (1) and (4), and the inequality $h(x) \leq f(x)$ for $x \in [a, c] \cup [a + b - c, b]$, we get

$$\frac{1}{2} f(c) + \frac{1}{2} f(a + b - c) = \frac{1}{2} h(c) + \frac{1}{2} h(a + b - c)$$

$$= h\left(\frac{a + b}{2}\right) = h\left(\frac{\int_a^c x\,dx + \int_{a+b-c}^b x\,dx}{2(c - a)}\right)$$

$$= \frac{\int_a^c h(x)dx + \int_{a+b-c}^b h(x)dx}{2(c - a)} \leq \frac{\int_a^c f(x)dx + \int_{a+b-c}^b f(x)dx}{2(c - a)}. \quad (30)$$

Continuing with Equation (30) by applying the Hermite-Hadamard inequality and the inequality in Equation (25), it follows

$$\leq \frac{f(a) + f(c) + f(a + b - c) + f(b)}{4} \leq \frac{f(a) + f(b)}{2}. \quad (31)$$

It remains only to multiply by 2. □

Corollary 3.3. *Let* $[a, b]$ *be a closed interval of real numbers, and let* $[c, d] \subset (a, b)$ *be a closed subinterval satisfying the common barycenter condition in Equation* (16).

Then every convex function $f : [a, b] \to \mathbb{R}$ *satisfies the double integral inequality*

$$\frac{\int_c^d f(x)dx}{d - c} \leq \frac{\int_a^b f(x)dx}{b - a} \leq \frac{\int_a^c f(x)dx}{2(c - a)} + \frac{\int_d^b f(x)dx}{2(b - d)}. \quad (32)$$

Proof. Midpoint equality in Equation (16) can be expressed in the integral form by the barycenter equalities

$$\frac{\int_c^d x\,dx}{d-c} = \frac{\int_a^b x\,dx}{b-a} = \frac{\int_a^c x\,dx}{2(c-a)} + \frac{\int_d^b x\,dx}{2(b-d)}. \tag{33}$$

We firstly prove that the left term of Equation (32) is less than or equal to the right term. Let $y = h(x)$ be the secant line of the convex curve $y = f(x)$ passing through the graph points $C(c, f(c))$ and $D(d, f(d))$. Using inequalities $f(x) \le h(x)$ for $x \in [c, d]$, and $h(x) \le f(x)$ for $x \in [a, c] \cup [a+b-c, d]$, and applying the affinity of function h to Equation (33), we get

$$\frac{\int_c^d f(x)dx}{d-c} \le \frac{\int_c^d h(x)dx}{d-c} = \frac{\int_a^c h(x)dx}{2(c-a)} + \frac{\int_d^b h(x)dx}{2(b-d)}$$

$$\le \frac{\int_a^c f(x)dx}{2(c-a)} + \frac{\int_d^b f(x)dx}{2(b-d)}. \tag{34}$$

Now, the inequality in Equation (32) can be confirmed by the combination

$$\frac{\int_a^b f(x)dx}{b-a} = \frac{\int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx}{b-a}$$

$$= \alpha\left[\frac{\int_c^d f(x)dx}{d-c}\right] + \beta\left[\frac{\int_a^c f(x)dx}{2(c-a)} + \frac{\int_d^b f(x)dx}{2(b-d)}\right], \tag{35}$$

which is convex because the coefficients

$$\alpha = \frac{d-c}{b-a},\ \beta = \frac{2(c-a)}{b-a} = \frac{2(d-b)}{b-a} \tag{36}$$

are nonnegative and their sum is equal to 1. □

Combining the inequalities in Equations (14), (32), (28), and (29), we get the following integral refinement of the Hermite-Hadamard inequality.

Theorem 3.4. *Let* $[a, b]$ *be a closed interval of real numbers, and let* $[c, d] \subset (a, b)$ *be a closed subinterval satisfying the common barycenter condition in Equation* (16). *Then every convex function* $f : [a, b] \to \mathbb{R}$ *satisfies the series of inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{\int_c^d f(x)dx}{d-c} \le \frac{\int_a^b f(x)dx}{b-a}$$

$$\le \frac{\int_a^c f(x)dx}{2(c-a)} + \frac{\int_d^b f(x)dx}{2(b-d)} \le \frac{f(a)+f(b)}{2}. \tag{37}$$

4. Application to Means

Thorough this section we use positive numbers a and b, and a strictly monotone continuous function $\varphi : [a, b] \to \mathbb{R}$.

The discrete quasi-arithmetic mean of the numbers a and b respecting the function φ can be defined by the number

$$M_{\varphi}(a, b) = \varphi^{-1}\left(\frac{1}{2}\varphi(a) + \frac{1}{2}\varphi(b)\right). \tag{38}$$

Using the identity function $\varphi(x) = x$, we get the generalized arithmetic mean

$$A(a, b) = \frac{1}{2}a + \frac{1}{2}b, \tag{39}$$

using the hyperbolic function $\varphi(x) = 1/x$, we have the generalized harmonic mean

$$H(a, b) = \left(\frac{1}{2}a^{-1} + \frac{1}{2}b^{-1}\right)^{-1}, \tag{40}$$

and using the logarithmic function $\varphi(x) = \ln x$, we obtain the generalized geometric mean

$$G(a, b) = a^{\frac{1}{2}} b^{\frac{1}{2}}. \tag{41}$$

The above means satisfy the generalized harmonic-geometric-arithmetic mean inequality

$$H(a, b) < G(a, b) < A(a, b). \tag{42}$$

The integral quasi-arithmetic mean of the numbers a and b respecting the function φ can be defined by the number

$$M_{\varphi}(a, b) = \varphi^{-1}\left(\frac{1}{b-a}\int_a^b \varphi(x)dx\right). \tag{43}$$

Using the hyperbolic function, we have the logarithmic mean

$$L(a, b) = \left(\frac{1}{b-a}\int_a^b \frac{1}{x}\,dx\right)^{-1} = \frac{b-a}{\ln b - \ln a}, \tag{44}$$

and using the logarithmic function, we obtain the identric mean

$$I(a, b) = \exp\left(\frac{1}{b-a}\int_a^b \ln x\,dx\right) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}. \tag{45}$$

The well-known mean inequality asserts that

$$H(a, b) < G(a, b) < L(a, b) < I(a, b) < A(a, b). \tag{46}$$

Applying Equation (18) to the convex function $f(x) = -\ln x$ using substitutions $a \mapsto 1/a,\ b \mapsto 1/b,\ c \mapsto 1/c,$ and $d \mapsto 1/d,$ and then acting on the rearranged inequality with the exponential function, we can derive the series of inequalities

$$H(a, b) \leq \frac{H(a, c)H(c, d)H(d, b)}{32}$$

$$\leq \frac{I^{-1}(a^{-1}, c^{-1})I^{-1}(c^{-1}, d^{-1})I^{-1}(d^{-1}, b^{-1})}{32}$$

$$\leq (ac^3d^3b)^{1/8} \leq G(a, b), \tag{47}$$

refining the harmonic-geometric mean inequality.

Applying Equation (37) to the exponential function $f(x) = e^x$ using substitutions $a \mapsto \ln a$, $b \mapsto \ln b$, $c \mapsto \ln c$, and $d \mapsto \ln d$, we obtain the series of inequalities

$$G(a, b) \leq L(c, d) \leq L(a, b)$$

$$\leq \frac{1}{2} L(a, c) + \frac{1}{2} L(d, b) \leq A(a, b), \tag{48}$$

refining the geometric-logarithmic-arithmetic mean inequality.

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