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BETWEEN DEMICONTRACTIVE, α-DEMICONTRACTIVE MAPPINGS AND THE STRONG CONVERGENCE OF THE MANN ITERATIVE SEQUENCE IN HILBERT SPACES

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Abstract

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a demicontractive map satisfying $\langle Tx, x\rangle \geq \|x\|^2 - \lambda\|x - Tx\|^2$ for some $\lambda > 0$, then T is α-demicontractive. Furthermore, the Mann iterative sequence given by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, where $\alpha_n \in (0, 1)\ \forall n \geq 0$, converges strongly to an element of $F(T) := \{x \in C : Tx = x\}$.

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1. Introduction

Let H be a real Hilbert space. A mapping $T : H \to H$ is said to be demicontractive if there exists a constant $k > 0$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + k\|x - Tx\|^2, \tag{1.1}$$

for all $(x, p) \in H \times F(T)$, where $F(T) := \{x \in H : Tx = x\} \neq \emptyset$. More often than not, k is assumed to be in the interval $(0, 1)$. However, this is a restriction of convenience. If $k = 1$, then T is called a hemicontractive map.

On the other hand, T is said to satisfy condition (A) if there exists $\lambda > 0$ such that

$$\langle x - Tx, x - p\rangle \geq \lambda\|x - Tx\|^2, \tag{1.2}$$

for all $(x, p) \in H \times F(T)$.

The above classes of maps were studied independently by Hicks and Kubicek [3] and Maruster [5]. It is however shown in [1] that the two classes of maps coincide if $k \in (0, 1)$ and $\lambda \in (0, \frac{1}{2})$.

The class of demicontractive maps includes the class of quasi-nonexpansive and the class of strictly pseudocontractive maps. Any strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.

If C is a closed convex subset of any Banach space E and $T : C \to C$ is any map, then the Mann iteration sequence [2] is given by $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, where $\alpha_n \in (0, 1)\ \forall n \geq 0$, satisfying certain conditions. Several authors (see, e.g., [2], [3], [4], [5], [6]) have studied the convergence of the Mann iteration sequence to fixed points of certain mappings in certain Banach spaces. However, the Mann iteration sequence is very suitable for the study of convergence to fixed points of demicontractive mappings. It is well known (see, e.g., [1]) that demicontractivity alone is not sufficient for the convergence of the Mann iteration sequence. Some additional smoothness properties of T such as continuity and demiclosedness are necessary.

A map T is said to be demiclosed at a point x_0 if whenever $\{x_n\}$ is a sequence in the domain of T such that $\{x_n\}$ converges weakly to $x_0 \in D(T)$ and $\{Tx_n\}$ converges strongly to y_0, then $Tx_0 = y_0$.

Definition 1. Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$ and let C be a nonempty closed convex subset of H. The orthogonal projection $P_C x$ of x onto C is defined by $P_C x = \arg\min_{y \in C} \|x - y\|$, and has the following properties:

(i) $\langle x - P_C x, z - P_C x \rangle \leq 0$, for all $z \in C$.

(ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$, for all $x, y \in H$.

In [7], Maruster studied the convergence of the Mann iteration sequence for demicontractive maps, in finite dimensional spaces, with an application to the study of the so-called relaxation algorithm for the solution of a particular convex feasibility problem. More precisely, he proved the following:

Theorem 1 ([7]). *Let* $T : \Re^m \to \Re^m$ *be a nonlinear mapping, where* $\Re^m$ *is the m-Euclidean space. Suppose the following are satisfied*:

(i) $I - T$ *is demiclosed at* 0.

(ii) T *is demicontractive with constant* k, *or equivalently* T *satisfies condition A with* $\lambda = \frac{1-k}{2}$. *Then, the Mann iteration sequence converges to a point of* $F(T)$ *for any starting* x_0.

Maruster and Maruster [1] noted that in infinite dimensional spaces, demicontractivity and demiclosedness of T are not sufficient for strong convergence. However, the two conditions ensure weak convergence. More precisely, he proved the following:

Theorem 2 ([**5**]). *Let* $T : C \to C$ *be a nonlinear mapping with* $F(T) \neq \emptyset$, *where* C *is a closed convex subset of a real Hilbert space* H. *Suppose the following conditions are satisfied*:

(i) $I - T$ *is demiclosed at* 0.

(ii) T *is demicontractive with constant* k, *or equivalently* T *satisfies condition A with* $\lambda = \frac{1-k}{2}$.

(iii) $0 < a \leq \alpha_n \leq b < 2\lambda = 1 - k$.

Then the Mann iteration sequence converges weakly to a fixed point of $F(T)$, *for any starting* x_0.

2. Strong Convergence

As noted above, demicontractivity and demiclosedness of T are not sufficient for strong convergence of the Mann iteration sequence in infinite dimensional spaces. Some additional conditions on T, or some modifications of the Mann iteration sequence are required for strong convergence to fixed points of demicontractive maps. Such additional conditions or modifications have been studied by several authors (see, e.g., [3], [4], [6], [8], [9]).

There is however an interesting connection between the strong convergence of the Mann iteration sequence to a fixed point of a demicontractive map, T, and the existence of a nonzero solution of a certain variational inequality. This connection was observed by Maruster [5], and has been studied by several authors. More precisely, Maruster proved the following theorem:

Theorem 3 ([**5**]). *Suppose T satisfies the conditions of Theorem 2. If in addition there exists $0 \neq h \in H$ such that*

$$\langle x - Tx, h \rangle \leq 0, \tag{1.3}$$

for all $x \in D(T)$, then starting from a suitable x_0, the Mann iteration sequence converges strongly to an element of $F(T)$.

The conditions of/and the variational inequality in Theorem 3 have been used and generalized by several authors (see, e.g., [8], [9]). The existence of a nonzero solution to the variational inequality is sometimes gotten under very stringent conditions. In [1], Maruster and Maruster made the following observation "It would therefore be interesting to study more closely the existence of a nonzero solution of the variational inequality" (1.3). Furthermore in [1], the authors noted that if T satisfies the positivity type condition $\langle Tx, x \rangle \geq \|x\|^2$, then it is sufficient to find a nonzero solution of the variational inequality (1.3).

In order to overcome the difficulty surrounding the existence and nature of the nonzero solution of the variational inequality (1.3), Maruster and Maruster [1] recently introduced the class of α-demicontractive maps; where for a nonempty subset C of a Hilbert space, a map $T : C \to C$ is called α-demicontractive if there exists $\lambda > 0$ such that

$$\langle x - Tx, x - \alpha p \rangle \geq \lambda \|x - Tx\|^2, \tag{1.4}$$

for all $(x, p) \in C \times F(T)$ and some $\alpha \geq 1$. It is easy to see that (1.4) is equivalent to

$$\|Tx - \alpha p\|^2 \leq \|x - \alpha p\|^2 + k\|x - Tx\|^2, \tag{1.5}$$

for all $(x, p) \in C \times F(T)$ and $k > 0$. With this, Maruster and Maruster [1] proved the following strong convergence theorem:

Theorem 4. *Suppose T satisfies the conditions of Theorem* 2. *Suppose also that T is* α-*demicontractive*, *for some* $\alpha > 1$. *Then starting from a suitable* x_0, *the Mann iteration sequence converges strongly to a point in* $F(T)$.

It is our purpose in this paper is to provide a mild condition under which a demicontractive map is also α-demicontractive, and hence prove strong convergence of the Mann iteration sequence to a fixed point of *T*. The condition is embodied in the following lemma:

Lemma 1.1. *Let C be a nonempty subset of a Hilbert space*, *H*, *and let* $T : C \to C$ *be a demicontractive map*. *If in addition T satisfies* $\langle Tx, x\rangle \geq \|x\|^2 - \lambda\|x - Tx\|^2$ *for all* $x \in D(T)$, $\lambda > 0$ *being the constant in* (1.2), *then T is* α-*demicontractive.*

Proof. From the demicontractivity of *T* and the condition in the lemma, we have

$$\langle x - Tx, x - p\rangle \geq \lambda\|x - Tx\|^2 \ \forall x \in D(T), p \in F(T), \lambda > 0$$

$$\geq \|x\|^2 - \langle Tx, x\rangle$$

$$\Leftrightarrow \langle Tx, x\rangle - \langle x, x\rangle + \langle x - Tx, x - p\rangle \geq 0$$

$$\Leftrightarrow \langle Tx - x, x\rangle + \langle x - Tx, x - p\rangle \geq 0$$

$$\Leftrightarrow -\langle x - Tx, x\rangle + \langle x - Tx, x - p\rangle \geq 0$$

$$\Leftrightarrow \langle x - Tx, -x\rangle + \langle x - Tx, x - p\rangle \geq 0$$

$$\Leftrightarrow \langle x - Tx, -p\rangle \geq 0$$

$$\Leftrightarrow \langle x - Tx, p\rangle \leq 0.$$

Demicontractive maps *T* abound for which $p \neq 0$ or for which $F(T)$ is not a singleton set. Now, for such maps and for some $\alpha > 1$, we have as in the proof of **Fact 2** of [1] that

$$\langle x - Tx,\ x - p\rangle - \lambda\|x - Tx\|^2 \geq (\alpha - 1)\langle x - Tx,\ p\rangle$$

$$\Leftrightarrow \langle x - Tx,\ x - p\rangle - (\alpha - 1)\langle x - Tx,\ p\rangle \geq \lambda\|x - Tx\|^2$$

$$\Leftrightarrow \langle x - Tx,\ x - p - (\alpha - 1)p\rangle \geq \lambda\|x - Tx\|^2$$

$$\Leftrightarrow \langle x - Tx,\ x - \alpha p\rangle \geq \lambda\|x - Tx\|^2.$$

Hence, such maps are α-demicontractive.

Example. Let $H = R$ (reals) and $C = [0, 2]$ be a nonempty closed convex subset of H. Define $T : C \to C$ by

$$Tx = \begin{cases} 1, & \text{if } x < 1, \\ 2, & \text{if } 1 \leq x \leq 2. \end{cases}$$

Then T is demicontractive and satisfies the condition of our lemma. To see this, observe that $F(T) = \{2\}$ and

(i) For $0 \leq x < 1$, we have

$$\begin{aligned} \langle x - Tx,\ x - p\rangle &= \langle x - 1,\ x - 2\rangle \\ &= \langle x - 1,\ x - 1\rangle + \langle x - 1,\ -1\rangle \\ &\geq \langle x - 1,\ x - 1\rangle \text{ since } \langle x - 1,\ -1\rangle \geq 0 \\ &= \|x - Tx\|^2 \\ &\geq \lambda\|x - Tx\|^2 \ \forall\ 0 < \lambda < 1. \end{aligned}$$

Therefore, T is demicontractive. Also,

$$\langle Tx,\ x\rangle = \langle 1,\ x\rangle$$

$$= x \geq x^2 = \|x\|^2 \geq \|x\|^2 - \lambda\|x - Tx\|^2 \ \forall\ 0 < \lambda < 1.$$

Hence T satisfies the condition of our lemma.

(ii) For $1 \leq x \leq 2$, we have

$$\begin{aligned}\langle x - Tx,\ x - p\rangle &= \langle x - 2,\ x - 2\rangle \\ &= \|x - Tx\|^2 \\ &\geq \lambda\|x - Tx\|^2 \ \ \forall\, 0 < \lambda < 1.\end{aligned}$$

Therefore, T is demicontractive. Also,

$$\begin{aligned}&\langle Tx,\ x\rangle = \langle 2,\ x\rangle \\ &\quad = 2x \geq x^2 = \|x\|^2 \geq \|x\|^2 - \lambda\|x - Tx\|^2 \ \ \forall 0 < \lambda < 1.\end{aligned}$$

Hence, T satisfies the condition of our lemma.

Theorem 5. *Suppose T satisfies*:

(i) *The conditions of Theorem* 2.

(ii) $\langle Tx,\ x\rangle \geq \|x\|^2 - \lambda\|x - Tx\|^2$ *for all* $x \in D(T)$, $\lambda > 0$. *Then starting from a suitable* x_0, *the Mann iteration sequence converges strongly to an element of* $F(T)$.

Proof. From Lemma 1.1, T is α-demicontractive and the proof follows from Theorem 4.

Remark 1. In [1], Maruster and Maruster discussed the ways of choosing x_0. One other way of choosing x_0 is as follows: For any $\beta > 1$, choose $x_0 = P_C(\beta p)$, where $p \in F(T)$ and $P : H \to C$ is the metric projection. This follows since it is well-known (see Definition 1) that P is firmly nonexpansive (i.e., satisfies condition (ii) of Definition 1), so that

$$\begin{aligned}\|x_0 - p\|^2 &= \|P_C(\beta p) - P_C(p)\|^2 \\ &\leq \langle P_C(\beta p) - P_C(p),\ \beta p - p\rangle \\ &= \langle x_0 - p,\ (\beta - 1)p\rangle \\ &= (\beta - 1)\langle x_0 - p,\ p\rangle.\end{aligned}$$

This implies $\langle x_0 - p,\ p\rangle \geq \epsilon_0\|x_0 - p\|^2$, where $\epsilon_0 = \dfrac{1}{\beta - 1}$.

Remark 2. In [1], Maruster and Maruster noted that if T satisfies the positivity type condition $\langle Tx, x\rangle \geq \|x\|^2$, then it is sufficient to find a non-zero solution of the variational inequality (1.3). This motivates the condition in our lemma. Observe that any mapping that satisfies the positivity condition in [1] also satisfies the condition in our lemma. Therefore, the class of maps that satisfy our condition is wider than the class of maps that satisfy the positivity condition and hence our condition is weaker than the positivity condition of Maruster and Maruster [1].

Remark 3. The class of demicontractive maps that satisfy our monotonicity condition is non-void (see example above).

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