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DEFORMATION OF A CLASS OF NON-COMPACT SPECIAL LAGRANGIAN SUBORBIFOLDS

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Abstract

The theory of strongly asymptotically conical special Lagrangian submanifolds and compact special Lagrangian suborbifolds have been developed by Marshall [12] and Zhang [21], respectively. In this note, we combine their methods to study the deformation of non-compact special Lagrangian suborbifolds.

1. Introduction

As a very interesting extension of deformation theory for complex submanifolds, Mclean [14] developed the deformation theory of special Lagrangian submanifolds, which have become important because Strominger et al. [19, 20] related the moduli space of special Lagrangian toric with flat unitary line bundle to the context of mirror symmetry. The

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theory is generalized to various situations ([1, 3, 12, 17, 18]) in the last few years. For the study of non-compact special Lagrangian submanifolds, Joyce presented several results in his series paper ([6]-[10]) and Pacini [15] considered the asymptotically conical special Lagrangian submanifolds. In particular, Marshall [13] studied the deformation of strongly asymptotically conical special Lagrangian submanifolds of $\mathbb{C}^n$, and Zhang [21] generalized the theory by Mclean and Hitchin to the deformation of compact special Lagrangian suborbifolds in a special class of Calabi-Yau orbifolds. Our purpose is to combine their methods together to study the deformation of non-compact special Lagrangian suborbifolds in special case.

Let $(\widetilde{J}, \widetilde{\omega}, \widetilde{\Omega})$ be the standard Calabi-Yau structure on $\mathbb{C}^n$ with Kähler metric $\widetilde{e}$, and Γ be a finite group acting on $\mathbb{C}^n$ preserving the structure $(\widetilde{J}, \widetilde{\omega}, \widetilde{\Omega})$. Consider the Calabi-Yau orbifold $(M, J, \omega, \Omega) = (\mathbb{C}^n, \widetilde{J}, \widetilde{\omega}, \widetilde{\Omega})/\Gamma$. Let $C \subset \mathbb{C}^n$ be a cone, smooth away from 0, and Γ-invariant. An embedded special Lagrangian orbifold $f : X \to M$ (cf. Subsection 2.3), where X is a manifold with ends, is said to be *strongly asymptotically conical with cone C and rate* $\alpha + 1 < 1$, if there exists an embedded special Lagrangian submanifold $\widetilde{f} : X \to \mathbb{C}^n$, which is strongly asymptotically conical with cone C and rate $\alpha + 1 < 1$ (see Subsection 2.2 for the precise definition), such that $\Gamma \cdot \widetilde{f}(X) = \widetilde{f}(X)$ and $q \circ \widetilde{f} = f$, where $q : \mathbb{C}^n \to M$ is the natural projection. Moreover, for $k \in \mathbb{N}$ and $0 < a < 1$, we say f to be of class $C^{k,a}$ (resp., C^k), if $\widetilde{f}$ is of class $C^{k,a}$ (resp., C^k). Denote by $\mathcal{M}^{k,a}$ the set of all $C^{k,a}$ embedded special Lagrangian suborbifolds $f : X \to M$, which are strongly asymptotically conical with cone C and rate $\alpha + 1 < 1$. Denote by

$$\widetilde{\mathcal{M}}^{k+1,a}, \tag{1.1}$$

the set of all $C^{k+1,a}$-embedded special Lagrangian submanifolds $\widetilde{f} : X \to \mathbb{C}^n$, which are strongly asymptotically conical with cone C and rate $\alpha + 1 < 1$. Clearly, there exists a natural action of Γ on it and $\widetilde{\mathcal{M}}^{k+1,a} / \Gamma = \mathcal{M}^{k+1,a}$. We shall prove that $\widetilde{\mathcal{M}}^{k+1,a}$ is a manifold (and thus $\mathcal{M}^{k+1,a}$ is an orbifold). In order to the goal, we define a map $\mathfrak{F}_{\alpha+1}$ and prove that its derivative at $(0, 0)$ is an invertible operator. By applying the implicit function theorem, it is easy to show that $\widetilde{\mathcal{M}}^{k+1,a}$ is a manifold. Moreover, in order to prove that $\mathcal{M}^{k+1,a}$ is an orbifold we need to show that every $f \in \widetilde{\mathcal{M}}^{k+1,a}$ is Γ-invariant, which is given in Section 3. Here is our main result.

Theorem 1.1. *Under the above assumptions, let* $f : X \to M$ *be a* $C^{k+1,a} (k \geq 2)$ *embedded special Lagrangian suborbifold and strongly asymptotically conical with cone* C *and rate* $\alpha + 1 < 1$, *and let* $\widetilde{f} : X \to \mathbb{C}^n$ *be its corresponding* Γ*-invariant lift as above. Let* $\alpha + 2 > 2 - n - \lambda$ *with* $\alpha + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ *(see Section* 3 *for the precise definition). Define* $K_{\alpha+1,\Gamma}$ *to be the subspace of all* Γ*-invariant elements in*

$$K_{\alpha+1} := \{\xi \in C_{\alpha+1}^{k+1,a}(T^*X) : d^*\xi = 0, d\xi = 0\}.$$

Then there exist two orbifolds $\mathfrak{D}$ *and* $\mathfrak{P}$, *a point* $b_0 \in \mathfrak{P}$, *two orbifold maps* $G : \mathfrak{D} \to \mathfrak{P}$ *and* $\mathrm{ev} : \mathfrak{D} \to \mathcal{M}^{k+1,a}$ *such that*

(i) $\mathrm{ev}(G^{-1}(b_0)) = f$ *and the dimension of* $\mathfrak{P}$ *is equal to dimension of* $K_{\alpha+1,\Gamma}$.

(ii) *For any* $b \in \mathfrak{P}$, $\mathrm{ev}(G^{-1}(b)) : X \to \mathbb{C}^n/\Gamma$ *is a special Lagrangian suborbifold of* $\mathbb{C}^n/\Gamma$, *which is strongly asymptotically conical with cone C and rate* $\alpha + 1 < 1$.

2. Preliminaries

2.1. Analysis of non-compact manifolds

We here recall the analytic theory on non-compact manifolds given in [13]. Without special statements, we always assume that X is a non-compact manifold of dimension $n \geqslant 3$ and that Σ is a compact manifold of dimension $n - 1$ with L connected components $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_L$. We also suppose that there exists a compact submanifold with boundary $X_0 \subseteq X$ and a diffeomorphism

$$X_\infty := X \setminus X_0 \to (0, \infty) \times \Sigma, \tag{2.1}$$

This is, X is said to be a *manifold with ends*. The identification in (2.1) leads to a projection onto the link of the cylindrical part of X, $\pi : X_\infty \to \Sigma$. Let t denote the conical coordinate on $(0, \infty)$, and let $(x_2 \cdots x_n)$ denote the coordinates on Σ. For $S \geq 0$, put

$$X_S = X_0 \cup ((0, S] \times \Sigma).$$

It is a compact submanifold of X with boundary. Fixing any covering of Σ, $\{U_1, \cdots, U_N\}$, and writing $V_\nu := (0, \infty) \times U_\nu$ for each $\nu = 1, \cdots, N$, we get an open cover of X_∞, $\{V_1, \cdots, V_N\}$. (Hereafter, we often identify X_∞ with $(0, \infty) \times \Sigma$). Then fix any open covering of X_0, $\{V_{N+1}, \cdots, V_{N+K}\}$, such that

$$\bigcup_{\nu=N+1}^{N+K} V_\nu \subseteq X_1,$$

and also fix the partition of unity of X, $\rho_1, \cdots, \rho_{N+K}$, subordinate to the open cover $\{V_1, \cdots, V_{N+K}\}$.

Let $E_\Sigma \to \Sigma$ be the vector bundle, which is trivial over each U_ν. Then, we have induced trivializations for the vector bundle $\pi^* E_\Sigma \to X_\infty$ over each $V_1, \cdots, V_N$. Suppose that $E \to X$ is a vector bundle over X, trivialized over each V_ν, so that $E|_{X_\infty} = \pi^* E_\Sigma$ on $X \setminus X_S$ for some large $S \geq 0$. We call such a vector bundle E over X *admissible* and the vector bundle $E_\Sigma \to \Sigma$ the *slice* of E over Σ. For the section ξ of an admissible bundle E, we denote by $\xi_1^\nu, \cdots, \xi_{\mathrm{rank} E}^\nu$ the components of ξ in the given trivialization of E over V_ν.

Let E be an admissible vector bundle with slice E_Σ as above. The fibre metric $\widetilde{\langle | \rangle}_E$ on E is said to be *translation invariant*, if there exists a metric $\langle | \rangle_{E_\Sigma}$ on E_Σ such that

$$\pi^* \langle | \rangle_{E_\Sigma} = \widetilde{\langle | \rangle}_E,$$

over $X \setminus X_S$ for some large $S \geq 0$. Here are some examples of admissible bundles:

- The tensor bundles $E := (\otimes^r T^* X) \otimes (\otimes^s TX)$, which have slices

$$\oplus_{i=r, r-1 \; j=s, s-1} (\otimes^i T^* \Sigma) \otimes (\otimes^j T\Sigma).$$

- The exterior bundles $E := \Lambda^r T^* X$, which have slices $\Lambda^r T^* \Sigma \oplus \Lambda^{r-1} T^* \Sigma$.

- The total exterior bundle $E := \Lambda^* T^* X$, which have slices $\Lambda^* T^* \Sigma \oplus \Lambda^* T^* \Sigma$.

To see why the slices are as given, consider the example $E = \Lambda^* T^* X$. For any given $x \in X_\infty$ and any section $\xi \in \Lambda^* T_x^* X$, there are unique $\phi, \psi \in \Lambda^* T_\sigma^* \Sigma$ such that $\xi = \phi + dt \wedge \psi$, where $x = (t, \sigma) \in (0, \infty) \times \Sigma = X_\infty$.

In the following, we always assume that E is one of the three bundles above without special statements. Then a linear operator $e^{(s-r)t}$ acts on section ξ of E as follows. If ξ has r covariant (T^*X) parts and s contravariant (TX) parts $e^{(s-r)t}\xi$ is defined to be $f_{r,s}\xi$, where $f_{r,s} : X \to (0, \infty)$ is a smooth function, which over X_∞ is equal to the exponential function $e^{(s-r)t}$. Then extend the operator $e^{(s-r)t}$ by linearity to act on any section ξ of E. It is invertible.

Suppose that the manifold E_Σ is equipped with a Riemannian metric g_Σ. A metric $\widetilde{h}$ on X, which is of the form

$$\widetilde{h} = dt^2 + g_\Sigma,$$

over $X \setminus X_S$ for some large $S \geq 0$ is called a *cylindrical* metric on X. A metric h on X is said to be *asymptotically cylindrical*, if there exists a cylindrical metric $\widetilde{h}$ such that

$$\sup_{\{t\}\times U_\nu} |\rho_\nu \partial^\lambda (h_{ij} - \widetilde{h}_{ij})| = o(1),$$

for each $1 \leq \nu \leq N, 1 \leq i, j \leq n,$ and $|\lambda| \geq 0$. Such a metric is always complete, and induces an asymptotically translation invariant fibre metric on each of the above three kinds of the admissible bundles.

For $\beta = (\beta_1, \cdots, \beta_L) \in \mathbb{R}^L$, let βt express smooth functions $X \to \mathbb{R}$, which are equal to $\beta_j t$ on the j-th end $(0, \infty) \times \Sigma_j$ of X. We write $\beta < a$ (resp., $\beta \leq a$), if $\beta_j < a$ (resp., $\beta_j \leq a$) for $a \in \mathbb{R}$ and $j = 1, \cdots, L$.

Following [13, page 55], given an asymptotically cylindrical metric on X, we have a *damped* B^k-space

$$B^k_\beta(E) = \{\xi \in C^k(E) : \sup_{\{t\}\times\Sigma} |\nabla^j_h \xi|_h = O(e^{\beta t}), \forall 0 \leq j \leq k\},$$

whose complete norm is given by

$$\|\xi\|_k := \sum_{j=0}^{k} \sup_X |e^{-\beta t}\nabla_h^j \xi|_h, \quad \forall \xi \in B_\beta^k(E).$$

We have also a *damped* Hölder space

$$B_\beta^{k,a}(E) = \{\xi \in B_\beta^k(E) : [e^{-\beta t}\nabla_h^j \xi]_{a,X}^h < \infty\},$$

whose complete norm is given by

$$\|\xi\|_{k,a} := [e^{-\beta t}\nabla_h^j \xi]_{a,X}^h + \sum_{j=0}^{k} \sup_X |e^{-\beta t}\nabla_h^j \xi|_h,$$

where $[\cdot]_{a,X}^h$ is defined as

$$[\xi]_{a,X}^h := \sup\left\{\frac{|\xi_x - \xi_y|_E}{d_h(x, y)^a} : x, y \in X \text{ with } 0 < d_h(x, y) < \varepsilon\right\}.$$

As before, we assume that the manifold Σ has a Riemannian metric g_Σ. Define a *cone metric* on by

$$\tilde{g} = e^{2t}(dt^2 + g_\Sigma).$$

A metric g on X is said to be *asymptotically conical*, if there exists a conical metric $\tilde{g}$ on X_∞ such that

$$\sup_{\{t\}\times U_\nu} |\rho_\nu \partial^\lambda (g_{ij} - \tilde{g}_{ij})| = o(e^{2t}),$$

for each $1 \le \nu \le N, 1 \le i, j \le n,$ and $|\lambda| \ge 0$. Such a metric is always complete.

Now suppose that X is endowed with some asymptotically conical metric g, asymptotic to the conical metric $\tilde{g}$ on X. Then $h := e^{-2t}g$ is asymptotically cylindrical metric, asymptotic to the cylindrical metric $\tilde{h} := e^{-2t}\tilde{g}$. According to [13, page 64], we let $C_\beta^k(E)$ be the set of all C^k

sections of E, which are forced to decay at rate $O(e^{\beta t})$ on the infinite piece X_∞ of X, as measured using the asymptotically conical metric g on X. Then a C^k section ξ of E lies in $C^k_\beta(E)$, if $e^{(s-r)t}\xi \in B^k_\beta(E)$. So as a vector space, we have $C^k_\beta(E) := e^{(s-r)t}B^k_\beta(E)$. Given $\xi \in C^k_\beta(E)$ define the norm

$$\|\xi\|_{C^k_\beta(E)} := \|e^{(s-r)t}\xi\|_{B^k_\beta(E)},$$

which makes $C^k_\beta(E)$ into a Banach space because $B^k_\beta(E)$ is a Banach space and the map

$$e^{(s-r)t} : C^k_\beta(E) \to B^k_\beta(E),$$

is an isometric isomorphism. Similarly, we define $C^{k,a}_\beta(E) := e^{(s-r)t}B^{k,a}_\beta(E)$ as a vector space. Then

$$\|\xi\|_{C^{k,a}_\beta(E)} := \|e^{(s-r)t}\xi\|_{B^{k,a}_\beta(E)},$$

gives a complete norm on $C^{k,a}_\beta(E)$ too. Here is a version of "conical damped embedding theorem".

Theorem 2.1 ([13, Theorem 4.17])**.** *If* $\beta \le \delta$ *and* $k + a \ge l + b$, *then there are continuous embeddings*

$$C^{k+1}_\beta(E) \subseteq C^{k,a}_\beta(E) \subseteq C^{l,b}_\delta(E) \subseteq C^l_\delta(E) \quad \textit{and} \quad C^k_\beta(E) \subseteq C^l_\delta(E).$$

Proof. Our method is derived from the proof of [11, Theorem 4.8]. In view of the second conclusion in [13, Theorem 4.2], we have the sequence of continue maps

$$C^{k+1}_\beta(E) \xrightarrow{\;e^{(r-s)t}\;} B^{k+1}_\beta(E) \to B^{k,a}_\beta(E) \xrightarrow{\;e^{-(r-s)t}\;} C^{k,a}_\beta(E).$$

Since $e^{(s-r)t}$ are isomorphic maps, it follows $C^{k+1}_\beta(E) \subseteq C^{k,a}_\beta(E)$, and the other results can be proved in the same way.

2.2. Asymptotically conical submanifolds of $\mathbb{C}^n$

A cone is a nonempty closed subset $C \subseteq \mathbb{R}^{2n}$ such that $C \setminus \{0\} \to \mathbb{R}^{2n}$ is a smooth submanifold and $e^t \cdot C = C$ for all $t \in \mathbb{R}$. The Euclidean metric $\tilde{e}$ on $\mathbb{R}^{2n}$ endows the manifold $C \setminus \{0\}$ with a metric $\tilde{g}$. There is an isomorphism

$$i : \mathbb{R} \times \Sigma \to C \setminus \{0\} \subseteq \mathbb{R}^{2n},$$

$$(t, \sigma) \mapsto e^t \sigma.$$

Using the identification $X \setminus X_0 \cong (0, \infty) \times \Sigma$, we can extend the restricted map $i : (0, \infty) \times \Sigma \to \mathbb{R}^{2n}$ to a smooth map $i : X \to \mathbb{R}^{2n}$.

For a map $\tilde{f} : X \to \mathbb{R}^{2n}$, if its components $\tilde{f}_1, \cdots, \tilde{f}_{2n} : X \to \mathbb{R}$ all lie in $C^k_\beta(X)$, then we write $\tilde{f} \in C^k_\beta(X, \mathbb{R}^{2n})$. It is easy to see that $i \in C^\infty_1(X, \mathbb{R}^{2n})$.

Let $\tilde{\alpha} \in \mathbb{R}^L$ with $\tilde{\alpha} < 1$. We call a submanifold $\tilde{f} : X \to \mathbb{R}^{2n}$ *strongly asymptotically conical* with cone C and rate $\tilde{\alpha}$, if $\tilde{f} - i \in C^\infty_{\tilde{\alpha}}$ $(X, \mathbb{R}^{2n})$. This is equivalent to the following condition:

$$\sup_{\{t\} \times \Sigma} |\nabla^j_{\tilde{g}} (\tilde{f}_k - i_k)|_{\tilde{g}} = O(e^{(\tilde{\alpha} - j)t}) \text{ for all } j \geq 0, \quad 1 \leq i \leq 2n.$$

Further assume that the submanifold $\tilde{f} : X \to \mathbb{C}^n$ is special Lagrangian and strongly asymptotically conical with cone $C \subseteq \mathbb{C}^n$ and the rate $\alpha + 1 < 1$, then C is also special Lagrangian submanifold by [13, Corollary 6.32].

2.3. The special Lagrangian suborbifolds

An n-dimensional orbifold is a paracompact Hausdorff space Y with an open covering $\mathcal{U} = \{U_i\}$ satisfying the following conditions:

(i) $\forall U_i, U_j \in \mathcal{U}, \exists U_k \in \mathcal{U}$ such that $U_k \subseteq U_i \cap U_j$ if $U_i \cap U_j \neq \emptyset$.

(ii) $\forall U_i \in \mathcal{U}$, there are a pair (V_i, Γ_i) consisting of a finite group Γ_i and a Γ_i-invariant open neighbourhood V_i of $0 \in \mathbb{R}^n$, and a Γ_i-invariant surjective continuous map $\tilde{\varphi} : V_i \to U_i$ that induces a homeomorphism $V_i / \Gamma_i \approx U_i$.

(iii) If $U_i \subseteq U_j$, then there exists an injection $\psi_{ij} : \Gamma_i \to \Gamma_j$, and an embedding $\phi_{ij} : V_i \to V_j$, which is equivariant with respect to ψ_{ij} (i.e., $\phi_{ij}(\gamma \cdot y) = \psi_{ij}(\gamma) \cdot \phi_{ij}(y)\ \forall y \in V_i, \gamma \in \Gamma)$ such that $\tilde{\varphi}_i = \tilde{\varphi}_j \circ \phi_{ij}$.

In an obvious way, one may define Riemannian orbifolds and complex orbifolds. In particular, a Kähler orbifold is a triple (Y, J, g) consisting of a complex orbifold (Y, J) and a Kähler metric g on it. (This means that g is J-invariant, i.e., $g(J\xi_1, J\xi_2) = g(\xi_1, \xi_2)\ \forall \xi_1, \xi_2 \in TM$, and that $\omega_g(\xi_1, \xi_2) := \frac{1}{2} g(J\xi_1, \xi_2)$ defines a closed non-degenerate 2-form, called the associated Kähler form on Y.) See [5, Subsection 6.5.1] for details. An *orbifold Calabi-Yau structure* on a Kähler orbifold (Y, J, g) is a triple (J, g, Ω), where Ω is a holomorphic volume form that satisfies $\nabla_g \Omega = 0$ for the Levi-Civita connection ∇_g and

$$(-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2} \right)^n \Omega \wedge \overline{\Omega} = \frac{1}{n!} \omega_g^n.$$

For $k \in \mathbb{N} \cup \{\infty\}$, a C^k map F from orbifolds Y to Z is said to be a C^k *immersion* (resp., *embedding*) if for each $y \in Y$, there is a chart (V_y, Γ_y) of Y, a chart $(V'_{F(y)}, \Gamma'_{F(y)})$ of Z, such that its local representation

$F_y : V_y \to V'_{F(y)}$ is an immersion (resp., embedding and the associated group homomorphism $\psi_y : \Gamma_y \to \Gamma'_{F(y)}$ is an isomorphism). In this case, $F(Y)$ is called a C^k *suborbifold* (resp., embedded suborbifold). If each F_y is also special Lagrangian (equivalently, $F^*\omega = 0$ and $F^*(\mathrm{Im}\,\Omega) = 0$), we get the notions of C^k special Lagrangian suborbifolds and C^k special Lagrangian embedded suborbifolds.

3. The Proof of Theorem 1.1

Let $\widetilde{f} : X \to \mathbb{C}^n$ be as in Theorem 1.1. By the assumptions, the finite group Γ preserves the Calabi-Yau structure $(\widetilde{J}, \widetilde{\omega}, \widetilde{\Omega})$, and $\Gamma \cdot \widetilde{f} = \widetilde{f}$. Then Γ acts on $(\widetilde{f}(X), \widetilde{e}|_{\widetilde{f}(X)})$ isometrically. Since $\widetilde{f}$ is an isometric embedding, Γ (resp., the metric $\widetilde{e}$ on $\mathbb{C}^n$) induces a Γ-action on X (resp., the original metric g on X). Later, we shall understand the action of Γ on X without special statements. Hence, there exists a Γ-action on harmonic 1-form space $\mathcal{H}^1$ given by $\gamma \cdot \theta = \gamma^{-1,*}\theta$ for all $\gamma \in \Gamma$, $\theta \in \mathcal{H}^1$, which naturally gives rise to an action on $C^{k+1,a}(X, \mathbb{C}^n) \times \mathcal{H}^1$:

$$\gamma \cdot (\widetilde{f}, \theta) = (\gamma \cdot \widetilde{f}, \gamma^{-1,*}\theta).$$

Let $N \to X$ be the normal bundle of X in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. That is, for any $p \in X$, the fiber N_p is the normal space of $T_{\widetilde{f}(p)}\widetilde{f}(X)$ in $\mathbb{R}^{2n}$. In particular, we may take $N_p = (T_{\widetilde{f}(p)}\widetilde{f}(X))^{\perp}$. Since Γ preserves metric, $\gamma_*\xi_p \in N_{\gamma \cdot p}$ for all $p \in X$ and $\gamma \in \Gamma$. By the Hopf-Rinow theorem [4], the subset $\widetilde{f}(X) \subseteq \mathbb{R}^{2n}$ is complete as a metric space and is closed in $\mathbb{R}^{2n}$. Hence, there exists a Γ-invariant open neighbourhood $\widetilde{U} \subseteq N$ of the zero section such that

$$\exp|_{\widetilde{U}} : \widetilde{U} \to \mathbb{R}^{2n},$$

is diffeomorphism onto an open subset of $\mathbb{R}^{2n}$, which is also Γ-equivariant, i.e.,

$$\gamma \cdot \exp_{\widetilde{f}(p)} \xi_p = \exp_{\gamma \cdot \widetilde{f}(p)} \gamma_* \xi_p, \quad \forall (p, \xi_p) \in \widetilde{U}.$$

It follows that any normal vector field $\xi \in C^\infty(N)$ with values in $\widetilde{U}$ defines an embedded submanifold $\widetilde{f}_\xi : X \to \mathbb{R}^{2n}$ given by

$$p \mapsto \widetilde{f}_\xi(p) := \exp_{\widetilde{f}(p)} \xi_p, \tag{3.2}$$

which is not necessarily Γ -invariant.

Since $\widetilde{f}^*\omega = 0$, the complex structure $\widetilde{J}$ defines a vector bundle isomorphism

$$\widetilde{J} : N \to T(\widetilde{f}(X)) \to TX.$$

Moreover, the metric g on X gives rise to an isomorphism

$$\flat_g : TX \to T^*X.$$

Hence, we can identify normal bundle N with T^*X via the composition $\flat_g \circ \widetilde{J}$.

Following [13, page 103], there exists a subset $\mathcal{D}(\Delta_g^0) \subset \mathbb{R}^L$ such that the bounded linear map $\Delta_g^0 : C_{\alpha+2}^{k+2,a}(X) \to C_\alpha^{k,a}(X)$ is Fredholm when $\alpha + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$. Here $\mathcal{D}(\Delta_g^0) = \mathcal{D}(P_\infty)(P_\infty = e^{2t}\Delta_{\widetilde{g}}^0)$ is computed as in [13, Subsection 5.1.1; see also Subsection 6.1.2].

Furthermore, according to [13, page 121], we assume that $\alpha + 2 > 2 - n - \lambda$ with $\alpha + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$, where the definition of λ is given in

[13, page 74], and choose $\beta_1 + 1, \beta_2 + 1 \in \mathbb{R}^L$ with $\beta_1 + 1 < \alpha + 1 < \beta_2 + 1 < 1$ and $\alpha - \beta_1 < n$ such that $\beta_1 + 2, \alpha + 2, \beta_2 + 2$ all belong to the same connected component of $\mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$. For any $\varepsilon > 0$, write

$$V_{\alpha+1}^{k+1,a} := \{\xi \in C_{\alpha+1}^{k+1,a}(T^*X) : \|\xi\|_{C_1^0(T^*X)} < \varepsilon\},$$

and define $F_{\alpha+1} : V_{\alpha+1}^{k+1,a} \to C^0(\Lambda^* T^* X)$ by

$$F_{\alpha+1}(\flat_g \widetilde{J}\xi) = *_g \widetilde{f}_\xi^* \operatorname{Im} \widetilde{\Omega} + \widetilde{f}_\xi^* \widetilde{\omega},$$

which actually takes values in $C_\alpha^{k,a}(X) \oplus C_\alpha^{k,a}(\Lambda^2 T^* X)$ by [13, Proposition 6.37]. Taking $\beta = \alpha$ in Propositions 6.38, 6.39, and 6.41 in [13], we get

Proposition 3.1 ([Proposition 6.39]). *Let $k \geq 2$ and $\alpha + 1 \in \mathbb{R}^L$ with $\alpha + 1 < 1$. Then the map $F_{\alpha+1} : V_{\alpha+1}^{k+1,a} \to C_\alpha^{k,a}(X) \oplus C_\alpha^{k,a}(\Lambda^2 T^* X)$ is smooth and has derivative*

$$F'_{\alpha+1}(0) : C_{\alpha+1}^{k+1,a}(T^*X) \to C_\alpha^{k,a}(X) \oplus C_\alpha^{k,a}(\Lambda^2 T^* X),$$

at 0 *which acts as $d^* + d$.*

Proposition 3.2 ([Proposition 6.41]). *Let $\alpha + 1 > 2 - n - \lambda$ with $\alpha + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$, then the image of map*

$$F_{\alpha+1} : V_{\alpha+1}^{k+1,a} \to C_\alpha^{k,a}(X) \oplus C_\alpha^{k,a}(\Lambda^2 T^* X),$$

is contained inside $d^*(C_{\alpha+1}^{k+1,a}(T^*X)) \oplus d(C_{\alpha+1}^{k+1,a}(T^*X))$.

Denote by $C_{\alpha+1}^{k+1,a}(\Lambda^* T^* X)_\Gamma$ the subspace of all Γ-invariant elements in $C_{\alpha+1}^{k+1,a}(\Lambda^* T^* X)$, and by $C_\alpha^{k,a}(X)_\Gamma$ the space of all Γ-invariant functions in $C_\alpha^{k,a}(X)$. Define

$$V^{k+1,a}_{\alpha+1,\Gamma} := V^{k+1,a}_{\alpha+1} \cap C^{k+1,a}_{\alpha+1}(\Lambda^* T^* X)_\Gamma,$$

$$K_{\alpha+1,\Gamma} := K_{\alpha+1} \cap C^{k+1,a}_{\alpha+1}(T^* X)_\Gamma,$$

where the set $K_{\alpha+1}$ is given in Theorem 1.1. Then $F_{\alpha+1}$ maps $V^{k+1,a}_{\alpha+1,\Gamma}$ into $d^*(C^{k+1,a}_{\alpha+1}(T^*X)_\Gamma) \oplus d(C^{k+1,a}_{\alpha+1}(T^*X)_\Gamma)$. Let

$$\mathfrak{B}_2 = \left\{\phi + \psi \in C^{k+1,a}_{\alpha}(X)_\Gamma \oplus C^{k+1,a}_{\alpha}(\Lambda^2 T^* X)_\Gamma : \phi \in \mathrm{Im}(d),\, \psi \in \mathrm{Im}(d^*)\right\},$$

and consider a map $\mathfrak{F}_{\alpha+1} : K_{\alpha+1,\Gamma} \times V^{k+1,a}_{\alpha+1,\Gamma} \to \mathfrak{B}_2$ defined by

$$\mathfrak{F}_{\alpha+1}(\xi_1, \xi_2) = *_g \tilde{f}_\xi^* \,\mathrm{Im}(\tilde{\Omega}) + \tilde{f}_\xi^* \tilde{\omega},$$

where $\xi = \xi_1 + \xi_2$. It is well-defined. In fact, as in the proof of [13, Proposition 6.41], there exist $\theta_1 \in C^\infty(T^*\mathbb{C}^n)$ and $\theta_{n-1} \in C^\infty(\Lambda^{n-1}T^*\mathbb{C}^n)$ such that $\mathrm{Im}\,\tilde{\Omega} = d\theta_{n-1}$ and $\tilde{\omega} = d\theta_1$. Then for $\xi_1 \in K_{\alpha+1,\Gamma}$, $\xi_2 \in V^{k+1,a}_{\alpha+1,\Gamma}$ and $\xi := \xi_1 + \xi_2$, we have

$$d^*\left\{(-1)^n * (\tilde{f}_\xi^* \theta_{n-1} - \tilde{f}^* \theta_{n-1})\right\} = *_g \tilde{f}_\xi^* \,\mathrm{Im}(\tilde{\Omega}),$$

$$d(\tilde{f}_\xi^* \theta_1 - \tilde{f}^* \theta_1) = \tilde{f}_\xi^* \tilde{\omega}, \quad \text{i.e.,} \quad \tilde{f}_\xi^* \tilde{\omega} - \tilde{f}^* \tilde{\omega} = \tilde{f}_\xi^* \tilde{\omega}.$$

Moreover, $\xi \in C^{k+1,a}_{\alpha+1}(T^*X)_\Gamma$ may be viewed as a Γ-invariant map $X \to \mathbb{C}^n$, i.e., $\xi \in C^{k+1,a}_{\alpha+1}(X, \mathbb{C}^n)$. Hence $\tilde{f}_\xi - \tilde{f} = \tilde{f} + \xi - \tilde{f} = \xi$ is Γ-invariant, and thus $\tilde{f}_\xi^* \tilde{\omega} = \tilde{f}_\xi^* \tilde{\omega} - \tilde{f}^* \tilde{\omega}$ is Γ-invariant. Note that strongly asymptotically is also asymptotically. We get $\tilde{f}, \tilde{f}_\xi \in C^{k+1,a}_1(X, \mathbb{C}^n)$, and hence $\tilde{f}_\xi^* \tilde{\omega} = \tilde{f}_\xi^* \tilde{\omega} - \tilde{f}^* \tilde{\omega} \in C^{k,a}_\alpha(\Lambda^2 T^* X)_\Gamma$ by [13, Proposition 6.31]. Similarly, we have $*_g \tilde{f}_\xi^* \,\mathrm{Im}(\tilde{\Omega}) \in C^{k,a}_\alpha(X)_\Gamma$. Thus, the image of $\mathfrak{F}_{\alpha+1}$ is contained in $\mathfrak{B}_2$.

By Proposition 3.1, $\mathfrak{F}_{\alpha+1}$ is smooth and has partial derivative at (0, 0),

$$D_{\xi_2}\mathfrak{F}_{\alpha+1}(0,\,0) : C^{k+1,\,a}_{\alpha+1}(\Lambda^* T^* X)_\Gamma \to \mathfrak{B}_2,$$

which acts as $d + d^*$ by Proposition 3.2. It follows from [13, Corollary 2.14] that $d + d^*$ is elliptic operator. Then by conical damped version of [13, Corollary 6.8], there exists a constant $C_1 > 0$ such that

$$\|\xi_2\|_{C^{k+1,\,a}_{\alpha+1}(\Lambda^* T^* X)_\Gamma} \le C_1 \|D_{\xi_2}\mathfrak{F}(0,\,0)\xi_2\|_{C^{k,\,a}_{\alpha}(\Lambda^* T^* X)_\Gamma}.$$

Thus $D_{\xi_2}\mathfrak{F}(0,\,0)$ is an invertible operator from $C^{k+1,\,a}_{\alpha+1}(T^* X)_\Gamma$ to $\mathfrak{B}_2$, and the invertible $D_{\xi_2}\mathfrak{F}(0,\,0)^{-1}$ is a bounded operator. Since $\tilde{f} : X \to \mathbb{C}^n$ is a special Lagrangian submanifold, $\mathfrak{F}(0,\,0) = 0$. By Theorem 2.1, we have $\|\xi_1\|_{C^0_1(T^* X)} \le C_2 \|\xi_1\|_{C^{k+1,\,a}_{\alpha+1}(T^* X)}$ for some constant C_2. Then the implicit function theorem ([16, Theorem 3.1]) implies that for $\varepsilon \ll 1$ and any $\|\xi_1\|_{C^{k+1,\,a}_{\alpha+1}(T^* X)} < \varepsilon / 2C_2$, there exists open neighbourhoods of the origin 0, $W_1^{\alpha+1} \subseteq K_{\alpha+1,\,\Gamma}$ and $\mathcal{W}_2^{\alpha+1} \subseteq V^{k+1,\,a}_{\alpha+1,\,\Gamma}$, and unique smooth map $\chi : W_1^{\alpha+1} \to \mathcal{W}_2^{\alpha+1}$ such that $\mathfrak{F}_{\alpha+1}(\xi_1,\, \chi(\xi_1)) = 0$ for all $\xi_1 \in W_1^{\alpha+1}$ and $\|\chi(\xi_1)\|_{C^0_1(T^* X)} < \varepsilon / 2C_2$. Then

$$\|\xi\|_{C^0_1(T^* X)} \le \|\xi_1 + \chi(\xi_1)\|_{C^0_1(T^* X)} \le \|\xi_1\|_{C^0_1(T^* X)} + \|\chi(\xi_1)\|_{C^0_1(T^* X)} < \varepsilon.$$

By [13, Theorem 6.43], we have $\xi \in V^{\infty}_{\alpha+1,\,\Gamma} \subseteq C^{\infty}_{\alpha+1}(T^* X)_\Gamma$. Moreover, every $\xi_1 \in W^{\alpha+1}$ gives a special Lagrangian submanifold $\tilde{f}_{\xi_1 + \chi(\xi_1)} : X \to \mathbb{C}^n$, which is strongly asymptotically conical with cone C and rate $\alpha + 1$.

Let $\widetilde{\mathcal{M}}^{k+1,a}$ be as in (1.1). Define

$$\widetilde{\mathfrak{D}} := \{ (\widetilde{f}, \xi_1) \in \widetilde{\mathcal{M}}^{k+1,a} \times K_{\alpha+1,\Gamma} : \| \xi_1 \|_{C^{k+1,a}_{\alpha+1}(T^*X)} < \varepsilon / 2C_2 \}.$$

Let π be the projection from $\widetilde{\mathfrak{D}}$ to

$$\widetilde{\mathfrak{P}} := \{ \xi_1 \in K_{\alpha+1,\Gamma} : \| \xi_1 \|_{C^{k+1,a}_{\alpha+1}(T^*X)} < \varepsilon / 2C_2 \},$$

and let $\widetilde{\mathrm{ev}} : \widetilde{\mathfrak{D}} \to \widetilde{\mathcal{M}}^{k+1,a}$ be given by

$$\widetilde{\mathrm{ev}}(\widetilde{f}, \xi_1) = \widetilde{f}_{\xi_1 + \chi(\xi_1)} =: \exp_{\widetilde{f}}(\xi_1 + \chi(\xi_1)).$$

Since the orbifold structure is Γ-invariant, and

$$\exp_{\widetilde{f}}(\gamma^{-1,*}\xi_1) = \gamma \cdot \exp_{\gamma^{-1}.\widetilde{f}} \xi_1,$$

with $\gamma^{-1,*} := (\gamma^{-1})^*$, we obtain

$$\begin{aligned}
&\mathfrak{F}_{\alpha+1}(\gamma^{-1,*}\xi_1, \gamma^{-1,*}\xi_2) \\
&\qquad = *_g \widetilde{f}^*_{\gamma^{-1,*}\xi} \operatorname{Im} \widetilde{\Omega} + \widetilde{f}^*_{\gamma^{-1,*}\xi} \widetilde{\omega} \\
&\qquad = \gamma^{-1,*}\{(\exp_{\gamma^{-1}.\widetilde{f}} \xi)^* \gamma^* \operatorname{Im} \widetilde{\Omega} + (\exp_{\gamma^{-1}.\widetilde{f}} \xi)^* \gamma^* \widetilde{\omega}\} \\
&\qquad = \gamma^{-1,*}\mathfrak{F}_{\alpha+1}(\xi_1, \xi_2),
\end{aligned}$$

for all $\gamma \in \Gamma$. So if $\mathfrak{F}_{\alpha+1}(\xi_1, \chi(\xi_1)) = 0$, then $\mathfrak{F}_{\alpha+1}(\gamma^{-1,*}\xi_1, \gamma^{-1,*}\chi(\xi_1)) = 0$. From the unique property of solution of $\mathfrak{F}_{\alpha+1}(\xi_1, \chi(\xi_1)) = 0$, it follows that $\xi_2 = \chi(\xi_1)$ is Γ-invariant, i.e., $\gamma^{-1,*}\chi(\xi_1) = \chi(\gamma^{-1,*}\xi_1)$, $\forall \gamma \in \Gamma$. This leads to

$$\widetilde{f}_{\gamma^{-1,*}\xi} = \mathrm{evp}_{\widetilde{f}}(\gamma^{-1,*}\xi_1 + \chi(\gamma^{-1,*}\xi_1)) = \mathrm{evp}_{\gamma.\widetilde{f}}(\gamma^{-1,*}\xi) = \gamma \cdot (\mathrm{evp}_{\widetilde{f}}\xi) = \gamma \cdot \widetilde{f}_\xi.$$

That is, if $\widetilde{f}_{\xi} : X \to \mathbb{C}^n$ is special Lagrangian so is $\widetilde{f}_{\gamma^{-1,*}\xi} : X \to \mathbb{C}^n$. Furthermore, we have the Γ-invariant of $\widetilde{\mathrm{ev}}$:

$$\widetilde{\mathrm{ev}}(\gamma \cdot (\widetilde{f}, \xi_1)) = (\widetilde{\gamma \cdot f})_{\gamma^{-1,*}\xi_1 + \chi(\gamma^{-1,*}\xi_1)}$$

$$= \mathrm{evp}_{\widetilde{f}}(\gamma^{-1,*}\xi_1 + \chi(\gamma^{-1,*}\xi_1))$$

$$= \gamma \cdot \widetilde{f}_{\xi} = \gamma \cdot \widetilde{\mathrm{ev}}(\widetilde{f}, \xi_1), \ \forall \gamma \in \Gamma.$$

Since $\widetilde{\mathrm{ev}}(\widetilde{f}, 0) = \widetilde{f} : X \to \mathbb{C}^n$ is a special Lagrangian submanifold, every $\xi_1 \in \widetilde{\mathfrak{P}}$ induces a special Lagrangian submanifold $\widetilde{\mathrm{ev}}(\widetilde{f}, \xi_1) = \widetilde{f}_{\xi_1 + \chi(\xi_1)} :$ $X \to \mathbb{C}^n$. Let

$$\mathfrak{D} = \widetilde{\mathfrak{D}} / \Gamma, \quad \mathfrak{P} = \widetilde{\mathfrak{P}} / \Gamma, \quad b_0 = 0 \in \mathfrak{P},$$

and let ev be the map from $\mathfrak{D}$ to $\mathcal{M}^{k+1,a}$ induced by $\widetilde{\mathrm{ev}}$, and an orbifold map $G : \mathfrak{D} \to \mathfrak{P}$ induced by π. Then $\mathrm{ev}(G^{-1}(0)) = f : X \to \mathbb{C}^n / \Gamma$ is a special Lagrangian suborbifold, and for any $b \in \mathfrak{P}$,

$$\mathrm{ev}(G^{-1}(b)) = \left(\coprod_{\xi_1 \in Q^{-1}(b)} \widetilde{f}_{\xi_1 + \chi(\xi_1)} \right) \Big/ \Gamma : X \to \mathbb{C}^n / \Gamma,$$

is a special Lagrangian suborbifold of the Calabi-Yau orbifold $(\mathbb{C}^n/\Gamma, J, e, \Omega)$, where $Q : \widetilde{\mathfrak{P}} \to \mathfrak{P}$ is the quotient map. □

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