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ON WEAKLY *s*-SEMIPERMUTABLE SUBGROUPS AND *p*-NILPOTENCY OF FINITE GROUPS*

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Abstract

In this paper, we investigate the p-nilpotency of a group for which every maximal subgroup of its Sylow p-subgroups is weakly s-semipermutable for some prime p. We get some results by new method and generalize some earlier results.

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1. Introduction

In this paper, all groups are finite and G stands for a finite group. Let $\pi(G)$ denote the set of all prime divisors of $|G|$. Let $\mathcal{F}$ denote a formation, $\mathcal{N}_p$ the class of all p-nilpotent groups and let us denote by $G^{\mathcal{F}} = \bigcap\{N \trianglelefteq G | G / N \in \mathcal{F}\}$ the $\mathcal{F}$-residual of G. "H Char G" means that H is a characteristic subgroup of G. The other notation and terminology are standard (see [6]).

Many authors have investigated the structure of a finite group when maximal subgroups of Sylow subgroups of the group are well situated in the group. Srinivasan [12] showed that a group G is supersolvable if all maximal subgroups of every Sylow subgroup of G are normal. Later, several authors obtain the same conclusion if normality is replaced by some weaker normality (see Chen [1]; Ramadan [8]; Skiba [11]); Wang [14]; Zhang [17]. Guo and Shum [5] proved the following result. Let p be an odd prime dividing $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent. Later on, Wang and Wang [13] get the same result by replacing the normality condition of maximal subgroups of Sylow subgroups by s-semipermutability. Moreover, if p is the smallest prime dividing $|G|$, then the assumption that $N_G(P)$ is p-nilpotent can be removed. These results have been particularly observed that the property of "normality" for some maximal subgroups of Sylow subgroups give a lot of useful information on the structure of groups.

In this paper, we investigate the p-nilpotency of a group for which every maximal subgroup of its Sylow p-subgroups is weakly s-semipermutable for some prime p. Some interesting results are obtained and many known results on this topic are generalized.

2. Basic Definitions and Preliminary Results

For two subgroups H and K of G, we say H permutes with K if $HK = KH$. We say, following Chen [1], a subgroup H of a group G is said to be s-semipermutable, or s-seminormal in G if it permutes with all Sylow p-subgroups P of G with $(p, |H|) = 1$. Recently, Xu and Li [15] introduced a new embedding property, namely, the weakly s-semipermutability of subgroups of a group.

Definition. A subgroup H of a group G is said to be weakly s-semipermutable in G if G has a subnormal subgroup T such that $HT = G$ and $H \cap T \leq H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the subgroup of H generated by all subgroups of H which are s-semipermutable in G.

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 ([17, Properties 1 and 2]). *Let G be a group and $A \leq H \leq G$. Then*:

(1) *If A is s-semipermutable in G, then A is s-semipermutable in H.*

(2) *Suppose that N is normal in G and A is a p-group. If A is s-semipermutable in G, then AN/N is s-semipermutable in G/N.*

Lemma 2.2 ([15, Lemma 2.3]). *Let G be a group and $A \leq E \leq G$. Then*:

(1) *If A is weakly s-semipermutable in G, then A is weakly s-semipermutable in E.*

(2) *Suppose that K is normal in G, and A is a p-group, $(|K|, p) = 1$. If A is weakly s-semipermutable in G, then AK/K is weakly s-semipermutable in G/K.*

Lemma 2.3 ([7, Lemma 6]). *Suppose that G is a non-abelian simple group. Then there exists an odd prime $r \in \pi(G)$ such that G has no Hall $\{2, r\}$-subgroup.*

Lemma 2.4 ([10, Lemma 1.6]). *Let P be a nilpotent normal subgroup of a group G. If* $P \cap \Phi(G) = 1$, *then P is the direct product of some minimal normal subgroups of G.*

3. Main Results and their Proofs

Theorem 3.1. *Suppose that N is a normal subgroup of a group G such that* G/N *is p-nilpotent and P is a Sylow p-subgroup of N, where* $p \in \pi(G)$ *with* $(|G|, p-1) = 1$. *If all maximal subgroup of P are weakly s-semipermutable subgroups of G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let G be a minimal counterexample with least $|N| + |G|$.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-nilpotent and $L \not\leq \Phi(G)$.

Let L be a minimal normal subgroup of G contained in N. Consider the factor group $\overline{G} = G/N$. Clearly $\overline{G}/\overline{N} \cong G/N$ is p-nilpotent and $\overline{P} = PL/L$ is a Sylow p-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Now let $\overline{P_1} = P_1L/L$ be a maximal subgroup of $\overline{P}$. We may assume that P_1 is a maximal subgroup of P. Then $P_1 \cap L = P \cap L$ is a Sylow p-subgroup of L. By the hypothesis, there is a subnormal subgroup B of G such that $G = P_1B$ and $P_1 \cap B \leq (P_1)_{\overline{s}G}$. We have $P_1L \cap BL = (P_1L \cap B)L$. Now we let $\pi(G) = \{p_1, p_2, \cdots, p_n\}$, where $p_1 = p$, and B_{p_i} be a Sylow p_i-subgroup of B $(i = 2, \cdots, n)$. Then B_{p_i} is also a Sylow p_i-subgroup of G, hence $B_{p_i} \cap N$ is a Sylow p_i-subgroup of N $(i = 2, \cdots, n)$. Write $V = \langle L \cap B_{p_2}, \cdots, L \cap B_{p_n} \rangle$, then $V \leq B$. Note that $(|L : P_1 \cap L|, |L : V|) = 1$, $L = (P_1 \cap L)V$, thus $P_1L \cap BL = (P_1L \cap B)L = (P_1V \cap B)L = (P_1 \cap B)VL = (P_1 \cap B)L$. It follows from Lemma 2.1 (2) that $(P_1L/L) \cap (BL/L)$

$= (P_1 \cap B)L / L \leq (P_1)_{\bar{s}G} L / L \leq (P_1 L / L)_{\bar{s}(G/L)}$. Therefore $\overline{P_1}$ is weakly s-semipermutable in $\overline{G}$. The choice of G implies that $\overline{G}$ is p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G contained in N and $L \not\leq \Phi(G)$.

(2) $O_p(N) = 1$.

If not, then by (1), $L \leq O_p(N)$ and, there is a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Since $M_p < P$, where $M_p \in Syl_p(M)$, we may let P_1 be a maximal subgroup of P containing M_p. Because P_1 is a weakly s-semipermutable subgroup of G, there exists a subnormal subgroup T of G such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{\bar{s}G}$. Since G / T_G is a p-group, we have $G / N \cap T_G$ is p-nilpotent. So $N \cap T_G \neq 1$ by the choice of G. Thus $L \leq N \cap T_G$. Furthermore, $(P_1)_{\bar{s}G}$ permutes with $T_q \in Syl_q(T) \subseteq Syl_q(G)$ for $p \neq q$, so $(P_1)_{\bar{s}G} T_q = T_q (P_1)_{\bar{s}G}$, thus $L \cap P_1 = L \cap P_1 \cap T = L \cap (P_1)_{\bar{s}G} = L \cap (P_1)_{\bar{s}G} T_q \lhd (P_1)_{\bar{s}G} T_q$, hence $T_q \leq N_G(L \cap P_1)$ for any $q \neq p$. Since $P \leq N_G(L \cap P_1)$, we have $L \cap P_1 \lhd G$. Thus $L \cap P_1 = L$ or $L \cap P_1 = 1$ by the minimal normality of L in G. If the former case is true, then $L \leq P_1$, so $P = LM_p = P_1$, a contradiction. Hence $L \cap P_1 = 1$. This means that L is cyclic of prime order. Hence G is p-nilpotent, a contradiction.

(3) End of the proof.

By (1) and (2), we get L is not solvable, then $L = S \times S \times \cdots \times S$, where S is a non-abelian simple group. Now, we claim that there exists a maximal subgroup P_1 of P such that $S_p \leq P_1$, where $S_p \in Syl_p(S)$. Assume that $P \cap L < P$, it is clear. So we may assume that $P \cap L = P$, then (L, L) satisfy the hypothesis by Lemma 2.2 (2). If L is not a non-abelian simple group, it is evident. If L is a non-abelian simple group, then every maximal subgroup of P is s-semipermutable in L. Suppose that P is cyclic, then L is p-nilpotent by Gorenstein [4, Theorem 6.3, p. 257]. This is a contradiction. Hence P has two different maximal subgroups, U and V say. Since U and V permutes with $L_q \in Syl_q(L)$ for $p \neq q$. Hence $PL_q = L_qP$ is a subgroup of L since $P = UV$. Therefore, P is s-semipermutable in L. We see that L is p-solvable by Chen [2, Theorem 2], a contradiction. So we can choose the maximal subgroup P_1 of P such that $S_p \leq P_1$. By the hypothesis, there is a subnormal subgroup B of G such that $G = P_1B$ and $P_1 \cap B \leq (P_1)_{\bar{s}G}$. Clearly, G / B_G is p-group, so $N \cap B_G \neq 1$. If not, then $G = G / N \cap B_G \lesssim G / N \times G / B_G$ is p-nilpotent, a contradiction. Thus $L \leq N \cap B_G$. For any $B_q \in Syl_q(B) \subseteq Syl_q(G)$ with $q \neq p$, we have $(P_1)_{\bar{s}G} B_q = B_q (P_1)_{\bar{s}G}$. Since $L \cap P_1B_q = L \cap (P_1B_q \cap B) = L \cap (P_1 \cap B)B_q \leq L \cap (P_1)_{\bar{s}G} B_q$, we get $L \cap P_1B_q = L \cap (P_1)_{\bar{s}G} B_q$, so $S \cap P_1B_q = S \cap (P_1)_{\bar{s}G} B_q$, thus $S \cap (P_1)_{\bar{s}G} = S \cap P_1 = S_p$ is a Sylow p-subgroup of S. Therefore, $S \cap (P_1)_{\bar{s}G} B_q$ is a Hall $\{p, q\}$-subgroup of S for any q with $q \neq p$. Since L is not solvable, we get $p = 2$ by the Odd Order Theorem. Hence, we have S is a non-abelian simple group with a Hall $\{2, q\}$-subgroup for any q with $q \neq 2$. This contradicts Lemma 2.3. We are done. □

Corollary 3.2. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If G is not p-nilpotent, then there is a maximal subgroup of $P \cap G^{\mathcal{N}_p}$, which is not weakly s-semipermutable in G.*

Corollary 3.3 ([5, Theorem 3.4])**.** *Let p be the smallest prime number dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 3.4 ([13, Theorem 3.3]). *Let p be the smallest prime number dividing the order of a group G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Corollary 3.5 ([16, Theorem 3.1.2]). *Suppose that N is a normal subgroup of a group G such that G / N is p-nilpotent and P is a Sylow p-subgroup of N, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If all maximal subgroup of P are weakly s-permutable subgroups of G, then G is p-nilpotent.*

Theorem 3.6. *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly s-semipermutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is not true and we choose G be a counterexample with the smallest order. Then we make the following claims:

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G / O_{p'}(G)$. By Lemma 2.2 (2), it is easy to see that $G / O_{p'}(G)$ satisfies the hypotheses of our theorem. Thus, by the minimality of G, we have $G / O_{p'}(G)$ is p-nilpotent, so is G, a contradiction.

(2) If M is a proper subgroup of G with $P \leq M < G$, then M is p-nilpotent.

It is clear that $N_M(P) \leq N_G(P)$ and hence $N_M(P)$ is p-nilpotent. Applying Lemma 2.2 (1), we see that M satisfies the hypotheses of our theorem. Now, by the minimality of G, M is p-nilpotent.

(3) $O_p(G) \neq 1$.

Since G is not p-nilpotent (p be an odd prime), we have $N_G(Z(J(P)))$ is not p-nilpotent by Glauberman-Thomposon Theorem, where $J(P)$ is a Thomposon subgroup of P. Clearly, $Z(J(P))$ char P, then $N_G(P) \leq N_G(Z(J(P)))$. If $N_G(Z(J(P))) < G$, by (2), $N_G(Z(J(P)))$ is p-nilpotent, a contradiction. So we may assume that $N_G(Z(J(P))) = G$, thus $O_p(G) \neq 1$.

(4) $G = PQ$, where Q is a Sylow q-subgroup of G with $q \neq p$.

Evidently, $G / O_p(G)$ is p-nilpotent and therefore, G is p-solvable. Then for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $PQ = QP$ is a subgroup of G by Gorenstein ([4, Theorem 6.3.5]). If $PQ < G$, then PQ is p-nilpotent by (2). It follows that $Q \leq C_G(O_p(G)) \leq O_p(G)$ by Robinson ([9, Theorem 9.3.1]) since $O_{p'}(G) = 1$, a contradiction. Thus, we have proven that $G = PQ$.

(5) Conclusion.

By (3), we can take a minimal normal subgroup L of G with $L \leq O_p(G)$. It is easy to see that the quotient group G / L satisfies the hypotheses of our theorem. Since the class of all p-nilpotent groups is a saturated formation, we may assume that L is the unique minimal normal subgroup of G and $L \nleq \Phi(G)$. Furthermore, by Lemma 2.4, we have that $O_p(G) = L$ is an elementary abelian p-group. Then there is a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Since $M_p < P$, where $M_p \in Syl_p(M)$, we may let P_1 be a maximal subgroup of P containing M_p. By the hypothesis, P_1 is a weakly s-semipermutable subgroup of G, so there exists a subnormal subgroup T such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{\bar{s}G}$. By the subnormality of T in G, we have $T_G \neq 1$, so $L \leq T_G$ by the unique minimal normality of L in G. Since $(P_1)_{\bar{s}G}$ permutes with $T_q \in Syl_q(T) \subseteq Syl_p(G)$ for $p \neq q$, we have $(P_1)_{\bar{s}G} T_q = T_q (P_1)_{\bar{s}G}$. Then $L \cap P_1 = L \cap P_1 \cap T = L \cap (P_1)_{\bar{s}G} = L \cap (P_1)_{\bar{s}G}$

$T_q \lhd (P_1)_{\bar{s}G} T_q$, so $T_q \leq N_G(L \cap P_1)$ for any $q \neq p$. Clearly, $P \leq N_G(L \cap P_1)$, we have $L \cap P_1 \lhd G$. Thus $L \cap P_1 = L$ or $L \cap P_1 = 1$ by the minimal normality of L in G. If the former case is true, then $L \leq P_1$, so $P = LM_p = P_1$, a contradiction. Hence $L \cap P_1 = 1$. Consequently, $|L| = p$, and therefore $Aut(L)$ is a cyclic group of order $p - 1$. If $p < q$, then LQ is clearly p-nilpotent and therefore $Q \leq C_G(L) = C_G(O_p(G))$, which contradicts to $C_G(O_p(G)) \leq O_p(G)$. If $q < p$, then, since $C_G(L) = C_G(O_p(G)) = O_p(G) = L$, we see that $G / L = N_G(L) / C_G(L) \lesssim Aut(L)$, so Q is a cyclic subgroup. Since Q is a cyclic and $q < p$, we know that G is q-nilpotent and therefore P is normal in G. Hence $N_G(P) = G$ is p-nilpotent, which is a contradiction. Thus, the proof of the theorem is complete. □

Corollary 3.7. *Let p be an odd prime dividing the order of a group G, P be a Sylow p-subgroup of G, and* $N_G(P)$ *be p-nilpotent. If G is not p-nilpotent, then there is a maximal subgroup of P, which is not weakly s-semipermutable in G.*

Corollary 3.8 ([5, Theorem 3.1])**.** *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If* $N_G(P)$ *is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 3.9 ([13, Theorem 3.1])**.** *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If* $N_G(P)$ *is p-nilpotent and every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Corollary 3.10 ([16, Theorem 3.1.3])**.** *Let p be a prime dividing the order of a group G and P be a Sylow p-subgroup of G. If* $N_G(P)$ *is p-nilpotent and every maximal subgroup of P is weakly s-permutable in G, then G is p-nilpotent.*

Proof. By Theorems 3.1 and 3.6, it is clear. □

Corollary 3.11. *Let N be a normal subgroup of a group G and p be an odd prime number dividing the order of N. Also, let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{N}_p$ *and* $G/N \in \mathcal{F}$. *Let P be a Sylow p-subgroup of N. If* $N_G(P)$ *is p-nilpotent and every maximal subgroup of P is weakly s-semipermutable in G, then* $G \in \mathcal{F}$.

Proof. The proof is very similar to the proof of [13, Corollary 3.2] and we omit it. □

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